

18.905 Problem Set 9

Due Wednesday, November 8 in class

1. Use naturality to show that if X is the disjoint union of subspaces X_α , then there is an isomorphism of rings

$$H^*(X) \rightarrow \prod_{\alpha} H^*(X_\alpha),$$

where the latter ring has the standard componentwise ring structure

$$(r_\alpha) \cdot (s_\alpha) = (r_\alpha s_\alpha).$$

2. Let X be the lens space $L(n, 1)$ from Hatcher, problem 8 on page 131. (We looked at this space on a previous assignment.) Compute (using the Δ -complex structure when necessary) the cohomology and cup product structure on $H^*(X; \mathbb{Z})$ and $H^*(X; \mathbb{Z}/n)$,
3. Suppose that C_* and D_* are chain complexes. Define a new chain complex $\underline{\text{Hom}}(C_*, D_*)$ which, in degree n , is

$$\underline{\text{Hom}}(C_*, D_*)_n = \prod_p \text{Hom}(C_p, D_{p+n}).$$

In other words, an element of this chain complex in degree n consists of a family of maps $f_p : C_p \rightarrow D_{p+n}$ (so n is the amount by which each map raises degree).

Give a definition of a boundary map $\delta : \underline{\text{Hom}}(C_*, D_*)_n \rightarrow \underline{\text{Hom}}(C_*, D_*)_{n-1}$ such that the evaluation map

$$\begin{aligned} \underline{\text{Hom}}(C_*, D_*) \otimes C_* &\rightarrow D_* \\ f \otimes x &\mapsto f(x) \end{aligned}$$

is a chain map, where the left-hand side has the standard Leibniz formula for its boundary.

What does it mean for an element of $\underline{\text{Hom}}(C_*, D_*)_0$ to be a cycle? When do two cycles in $\underline{\text{Hom}}(C_*, D_*)_0$ differ by a boundary?

4. Show the following dual version of the Künneth formula: If C_* is a chain complex where each C_p is levelwise free, and D_* is any chain complex, show that there are natural exact sequences

$$\begin{aligned} 0 \rightarrow \prod_{t-s=n+1} \text{Ext}(H_s(C_*), H_t(D_*)) \rightarrow H_n(\underline{\text{Hom}}(C_*, D_*)) \rightarrow \\ \prod_{t-s=n} \text{Hom}(H_s(C_*), H_t(D_*)) \rightarrow 0 \end{aligned}$$

which are split. (Hint: Follow the same method as the proofs of the Künneth formula and the universal coefficient theorems.)