18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

Operations on E_∞-Algebras: (Lecture 24)

In this lecture, we will change our notation in discussing E_∞-algebras: we will think of them as spectra, rather than complexes of $\mathbf{F}_2$-vector spaces. We will therefore view the cohomology of the underlying complex as a *homotopy group* of the underlying spectrum, so we have

$$\pi_n A \simeq \mathrm{H}^{-n} A.$$

We have constructed the big Steenrod algebra $\mathcal{A}^{\mathrm{Big}}$ so that it acts by stable operations on homotopy of E_∞-algebras over $\mathbf{F}_2$. Our goal in this lecture is to reformulate this in a more functorial way. Our discussion will be somewhat informal. To make these ideas precise, we really need to work in the setting of higher categories, but we will ignore this point.

We begin with some remarks about the category $\mathfrak{S}$ of spectra. For every pair of spectra X and Y, we have a smash product which we will denote by $X \otimes Y$. The smash product endows $\mathfrak{S}$ with a symmetric monoidal structure, and this symmetric monoidal structure is *closed*: that is, for every pair of spectra X and Y we can define a function spectrum $\mathrm{Map}(X, Y)$ with the following universal property:

$$\mathrm{Hom}(Z, \mathrm{Map}(X, Y)) = \mathrm{Hom}(X \otimes Z, Y).$$

The function spectrum $\mathrm{Map}(X, Y)$ has the property that $\pi_i \mathrm{Map}(X, Y)$ can be identified with the set of homotopy classes of maps from X into the i-fold suspension $Y[i]$.

In the special case where $X = Y$, the spectrum $\mathrm{Map}(X, X)$ is equipped with additional structure, given by composition of maps from X to X. The spectrum $\mathrm{Map}(X, X)$ is an example of an A_∞ (or associative) ring spectrum. It should be viewed as a ring of endomorphisms of X, and X is an example of a *module* over the spectrum $\mathrm{Map}(X, X)$.

More generally, for any category $\mathcal{C}$, the category of functors $\mathrm{Fun}(\mathcal{C}, \mathfrak{S})$ from $\mathcal{C}$ to spectra is *enriched* over spectra; that is, given a pair of functors $F, F' : \mathcal{C} \to \mathfrak{S}$, we can define a spectrum of maps $\mathrm{Map}(F, F')$. Again, in the special case where $F = F'$, we get an associative ring spectrum $R = \mathrm{Map}(F, F)$. For every object $C \in \mathcal{C}$, the spectrum $F(C)$ has a canonical action of the A_∞-algebra R.

We wish to study the special case in which $\mathcal{C}$ is the category of E_∞-algebras over $\mathbf{F}_2$, and the functor $G : \mathcal{C} \to \mathfrak{S}$ assigns to each E_∞-algebra A its underlying spectrum $G(A)$. The ring spectrum $R = \mathrm{Map}(G, G)$ then acts on the underlying spectrum of every E_∞-algebra A over $\mathbf{F}_2$, so that every element of $\pi_n R$ gives a map $A \to A[n]$, and therefore a map $\pi_k A \to \pi_{n+k} A$. This construction is functorial in A; we can therefore think of elements of $\pi_* R$ as giving rise to operations which act on the homotopy groups $\pi_* A$ for every E_∞-algebra A over $\mathbf{F}_2$.

Our goal in this lecture is to understand what the A_∞-algebra R looks like. More precisely, we will compute the homotopy groups of R and show that they coincide a suitably completion of the graded pieces of the big Steenrod algebra $\mathcal{A}^{\mathrm{Big}}$.

The forgetful functor G from E_∞-algebras over $\mathbf{F}_2$ to spectra can be described as a composition

$$\{E_\infty - \text{algebras over } \mathbf{F}_2\} \to \{\text{complexes over } \mathbf{F}_2\} \to \{\text{spectra}\}$$

where the first map forgets the multiplication, and the second map carries a complex V to the generalized Eilenberg-MacLane spectrum HV. This functor has a left adjoint $F : \mathfrak{S} \to \mathcal{C}$, given by the formula

$$F(X) = (\oplus_{n \geq 0} X^{\otimes n}_{h\Sigma_n}) \otimes \mathbf{F}_2.$$

Here the tensor product indicates the smash product of spectra, and we identify $\mathbf{F}_2$ with the Eilenberg-MacLane spectrum $H\mathbf{F}_2$. The adjointness between F and G yields a canonical identification of spectra

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathfrak{S})}(G, G) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, G \circ F).$$

We therefore need to be able to understand maps in the category of functors $\mathrm{Fun}(\mathfrak{S},\mathfrak{S})$ from spectra to spectra. To this end, we need to introduce a definition:

Definition 1. Let E be a functor from spectra to spectra. We say that E is *exact* if the following conditions are satisfied:

(1) The functor E carries zero objects to zero objects (i.e., if X is weakly contractible, then $E(X)$ is weakly contractible).

(2) For every spectrum X, the canonical map $\Sigma E(X) \to E(\Sigma X)$ (which exists in virtue of assumption (1)) is a weak homotopy equivalence.

Our calculation rests on the following observation:

Lemma 2. *Let E be an exact functor from spectra to spectra. Then the canonical map*

$$\alpha : \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E) \to \mathrm{Map}_{\mathfrak{S}}(S, E(S)) \simeq E(S)$$

is a weak equivalence. Here S denotes the sphere spectrum.

Sketch of proof. We will describe how to construct a map

$$E(S) \to \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E)$$

which is homotopy inverse to α. We can identify this with a collection of maps, $E(S) \otimes X \to E(X)$, depending functorially on the spectrum X.

Let X_n denote the nth space $\Omega^{\infty - n} X$ of the spectrum X, so we can identify X with the colimit of the sequence

$$\Sigma^\infty X_0 \to \Sigma^{-1}\Sigma^\infty X_1 \to \Sigma^{-2}\Sigma^\infty X_2 \to \dots.$$

We can identify $E(S) \otimes X$ with the colimit of the sequence $\Sigma^{-n}(E(S) \otimes \Sigma^\infty X_n)$, and we have a canonical map

$$\mathrm{colim}\, \Sigma^{-n} E(\Sigma^\infty X_n) \simeq \mathrm{colim}\, E(\Sigma^{-n}\Sigma^\infty X_n) \to E(X).$$

It therefore suffices to construct a compatible family of maps from $E(S) \otimes \Sigma^\infty X_n$ to $E(\Sigma^\infty X_n)$. Such a map is simply a map from X_n to the mapping space $[E(S), E(\Sigma^\infty X_n)]$, which arises by applying E to the canonical map from X_n to the mapping space $[*, X_n]$. □

Unfortunately, the composition $G \circ F \in \mathrm{Fun}(\mathfrak{S},\mathfrak{S})$ does not satisfy the hypotheses of Lemma 2. We have

$$(G \circ F)(X) \simeq \oplus_n (X^{\otimes n}_{h\Sigma_n}) \otimes \mathbf{F}_2;$$

in particular

$$(G \circ F)(0) \simeq \mathbf{F}_2.$$

To address this first obstruction, we have the following result:

Lemma 3. *Let E be a functor from spectra to spectra. For every spectrum X, the canonical map $X \to 0$ induces a map $E(X) \to E(0)$; let $E_0(X)$ denote the fiber of this map. Then the natural transformation $E_0 \to E$ induces a weak homotopy equivalence*

$$\alpha : \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E_0) \to \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E).$$

Sketch of proof. Let $Y = \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E)$, so that we have a canonical map $Y \otimes \mathrm{id} \to E$. Then, for every spectrum X, we get a commutative diagram

$$\begin{tikzcd} Y \otimes X \arrow[r] \arrow[d] & E(X) \arrow[d] \\ 0 \arrow[r] & E(0), \end{tikzcd}$$

which determines a map $Y \otimes X \to E_0(X)$. These maps together constitute a map $Y \to \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E_0)$, which is homotopy inverse to α. $\square$

Applying Lemma 3 to the composition $G \circ F$, we obtain the functor

$$X \mapsto (\oplus_{n>0} X^{\otimes n}_{h\Sigma_n}) \otimes \mathbf{F}_2.$$

This functor is still not exact. However, we can address the situation using Goodwillie's calculus of functors.

Lemma 4 (Goodwillie)**.** *Let E be a functor from spectra to spectra, and suppose that $E(0) \simeq 0$. Define a new functor $E' : \mathfrak{S} \to \mathfrak{S}$ by the formula*

$$E'(X) = \mathrm{proj\,lim}\{\ldots \to \Sigma^2 E(\Omega^2 X) \to \Sigma E(\Omega X) \to E(X)\}.$$

Then E' is exact, and the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E') \to \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E)$$

is a weak homotopy equivalence.

Proof. We will only prove the second statement. Since Σ^n and Ω^n are mutually inverse equivalences from the category of spectra to itself, we have canonical homotopy equivalences

$$\mathrm{Map}(\mathrm{id}, \Sigma^n \circ E \circ \Omega^n) \simeq \mathrm{Map}(\Omega^n \circ \mathrm{id} \circ \Sigma^n, E) \simeq \mathrm{Map}(\mathrm{id}, E).$$

The desired result now follows by passing to the limit. $\square$

We are now ready to compute the homotopy groups of the A_∞-algebra

$$R = \mathrm{Map}_{\mathrm{Fun}(\mathfrak{C},\mathfrak{S})}(G, G).$$

We first use Lemma 3 to replace $G \circ F$ by the pointed functor

$$E : X \mapsto \oplus_{n>0} X^{\otimes n}_{h\Sigma_n} \otimes \mathbf{F}_2,$$

and then Lemma 4 to replace E by its dual Goodwillie derivative E'. The functor E' is exact, and we have

$$\begin{aligned} R &= \mathrm{Map}_{\mathrm{Fun}(\mathfrak{C},\mathfrak{S})}(G, G) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, G \circ F) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E) \\ &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathfrak{S},\mathfrak{S})}(\mathrm{id}, E') \\ &\simeq E'(S) \\ &\simeq \mathrm{proj\,lim}\, \Sigma^k E(S^{-k}). \end{aligned}$$

It follows for every integer n, we have an exact sequence

$$0 \to \operatorname{proj}\overset{1}{\lim}\{\pi_{n+1-k}E(S^{-k})\} \to \pi_n R \to \operatorname{proj}\lim\{\pi_{n-k}E(S^{-k})\} \to 0.$$

We will show that the $\operatorname{proj}\lim^1$-term vanishes, and compute the limit on the right hand side.

By definition, $(G \circ F)(S^{-k})$ is the free E_∞-algebra $\mathcal{F}(k)$ on one generator in cohomological degree k, and $E(S^{-k})$ is its "augmentation ideal", so we have a canonical decomposition

$$\mathcal{F}(k) = E(S^{-k}) \oplus \mathbf{F}_2.$$

Therefore, we can identify $\pi_{n-k}E(S^{-k})$ with the summand of

$$F^{\mathrm{Big}}_{\mathrm{Alg}}(k)^{k-n} \simeq \mathbf{F}_2[\mathrm{Sq}^I \mu_k]^{k-n} :$$

spanned by those expressions of positive degree; here I ranges over all admissible sequences of integers having excess $< k$. Let us denote this summand by $\mathbf{F}_2[\mathrm{Sq}^I \mu_k]_0^{k-n}$.

We have an inverse system of graded vector spaces

$$\ldots \to \mathbf{F}_2[\mathrm{Sq}^I \mu_{k+1}] \xrightarrow{\theta_k} \mathbf{F}_2[\mathrm{Sq}^I \mu_k]_0 \to \ldots,$$

where each map θ_k lowers cohomological degrees by 1. Moreover, we have $\theta_k(\mu_{k+1}) = \mu_k$. Since the Steenrod operations are stable, it follows that $\theta_k(\mathrm{Sq}^I \mu_{k+1}) = \mathrm{Sq}^I \mu_k$. The map θ_k is induced by a map of E_∞-algebras

$$\mathcal{F}(k+1) \to \mathbf{F}_2 \times_{\mathcal{F}(k)} \mathbf{F}_2,$$

and the multiplication on the right hand side is trivial at the level of homotopy groups. It follows that θ_k vanishes on products.

The inverse system

$$\ldots \to \mathbf{F}_2[\mathrm{Sq}^I \mu_{k+1}] \xrightarrow{\theta_k} \mathbf{F}_2[\mathrm{Sq}^I \mu_k]_0 \to \ldots$$

is equivalent to the inverse system obtained by replacing each of the spaces $\mathbf{F}_2[\mathrm{Sq}^I \mu_k]_0$ by the image of θ_k. The above analysis shows that this subspace has a basis given by $\{\mathrm{Sq}^I \mu_k\}$, where I is an admissible sequence of integers having excess $\leq k$. We then obtain an inverse system of vector spaces

$$\ldots \to \mathbf{F}_2\{\mathrm{Sq}^I \mu_{k+1}\} \xrightarrow{\theta'_k} \mathbf{F}_2\{\mathrm{Sq}^I \mu_k\} \to \ldots$$

where the maps θ'_k are surjective. This proves the vanishing of the $\lim^1$-term, and shows that $\pi_n R$ is isomorphic to the inverse limit of the free vector spaces generated by the sets

$$\{\mathrm{Sq}^I \mu_k : I \text{ admissible of excess} \leq k \text{ and degree} = -n\}.$$

This vector space can be identified with a completion of $\mathcal{A}^{\mathrm{Big}\,-n}$. Recall that elements of $\mathcal{A}^{\mathrm{Big}}$ of degree $-n$ can be written as a finite sum

$$\sum_\alpha \mathrm{Sq}^{I_\alpha}$$

where I_α ranges over some collection of admissible sequences of integers which sum to $-n$. The vector space $\pi_n R$ is similar, except that we allow infinite sums

$$f = \mathrm{Sq}^{I_0} + \mathrm{Sq}^{I_1} + \ldots$$

so long as the excess of the sequences $\{I_k\}$ tends to ∞. (Note that, in this case, we can act by f on the cohomology of any E_∞-algebra A, since for each $x \in \mathrm{H}^n A$ almost all of the expressions $\mathrm{Sq}^{I_k} x$ will vanish by virtue of instability).