18.917 Topics in Algebraic Topology: The Sullivan Conjecture
Fall 2007

The Sullivan Conjecture (Lecture 30)

In this lecture we will combine some of our previous results to deduce a version of the Sullivan conjecture.

Theorem 1. *Let X be a finite-dimensional CW complex, $X^\vee$ its p-profinite completion, and K a connected p-profinite space. Then the diagonal map*

$$X^\vee \to (X^\vee)^K$$

is an equivalence of p-profinite spaces.

Proof. Let us say that a space X is *good* if $X^\vee \to (X^\vee)^K$ is an equivalence. Since p-profinite completion preserves homotopy pushout squares (being a left adjoint) and K is atomic in the p-profinite category, the collection of good spaces is stable under the formation of homotopy pushouts. We now show that every space X of finite dimension n is good, using induction on n. We have a homotopy pushout diagram

$$\begin{array}{ccc} \coprod S^{n-1} & \longrightarrow & \mathrm{sk}^{n-1} X \\ \downarrow & & \downarrow \\ \coprod D^n & \longrightarrow & X. \end{array}$$

The inductive hypothesis guarantees that $\mathrm{sk}^{n-1} X$ and $\coprod S^{n-1}$ are good. It will therefore suffice to show that $\coprod D^n$ is good. But this coproduct is homotopy equivalent to a discrete topological space, which is obviously good. □

Corollary 2. *Let X be a finite dimensional CW complex, and K a connected p-profinite space. Then every map $K \to X^\vee$ in the p-profinite category is homotopic to a constant map.*

In the special case where $K = BG$, where G is a finite p-group, we can identify $(X^\vee)^K$ with the homotopy fixed point set $(X^\vee)^{hG}$, where G acts trivially on X. There is a more general form of Theorem 1 where we do not assume that the action of G is trivial.

Lemma 3. *Let G be a finite p-group, and let $\mathfrak{S}_p^\vee(G)$ denote the category of p-profinite spaces with an action of G. Then the functor*

$$\mathfrak{S}_p^\vee(G) \to \mathfrak{S}_p^\vee$$

$$X \mapsto X^{hG}$$

preserves finite homotopy colimits.

Proof. We can identify $\mathfrak{S}_p^\vee(G)$ with $\mathfrak{S}_{p,/BG}^\vee$, and the formation of homotopy fixed points with the pushforward functor f_*, where $f : BG \to *$ is the projection. The desired result now follows from the observation that BG is atomic. □

Theorem 4. *Let G be a finite p-group, X a finite-dimensional G-CW complex, and X^G the subcomplex of G-fixed points. Then the composite map*

$$\phi : (X^G)^\vee \to (X^{hG})^\vee \to (X^\vee)^{hG}$$

is a homotopy equivalence of p-profinite spaces.

Proof. The space X admits a filtration

$$X^G = Y_{-1} \subseteq Y_0 \subseteq \ldots \subseteq Y_n = X,$$

where Y_j denotes the union of X^G with the j-skeleton of X. We will prove by induction on j that the conclusion of the theorem is valid for Y_j. The case $j = -1$ follows from Theorem 1. In the general case, we have a homotopy pushout diagram

$$\begin{array}{ccc} \coprod_\alpha S^{j-1} \times G/H_\alpha & \longrightarrow & Y_{j-1} \\ \downarrow & & \downarrow \\ \coprod_\alpha D^j \times G/H_\alpha & \longrightarrow & Y_j, \end{array}$$

where each H_α is a proper subgroup of G. Since p-profinite completion and passage to homotopy fixed points with respect to G preserve homotopy pushout squares, we get a homotopy pushout square

$$\begin{array}{ccc} ((\coprod_\alpha S^{j-1} \times G/H_\alpha)^\vee)^{hG} & \longrightarrow & (Y_{j-1}^\vee)^{hG} \\ \downarrow & & \downarrow \\ ((\coprod_\alpha D^j \times G/H_\alpha)^\vee)^{hG} & \longrightarrow & (Y_j^\vee)^{hG} \end{array}$$

of p-profinite spaces. By the inductive hypothesis, the upper right corner is equivalent to the p-profinite completion of X^G. It will therefore suffice to show that the p-profinite spaces in the left column are empty.

We will show that $Z = ((\coprod_\alpha S^{j-1} \times G/H_\alpha)^\vee)^{hG}$ is empty; the same argument will show that $((\coprod_\alpha D^j \times G/H_\alpha)^\vee)^{hG}$ is empty as well. The group G has only finitely many proper subgroups H. We can therefore decompose Z as a coproduct of spaces of the form

$$Z_H = ((\coprod_{H_\alpha = H} S^{j-1} \times G/H)^\vee)^{hG}.$$

It will therefore suffice to show that each Z_H is empty. But Z_H can be identified with

$$((\coprod S^{j-1})^\vee \times G/H)^{hG}.$$

We therefore have a map from Z_H to the homotopy fixed set $(G/H)^{hG}$, which is empty because H is a proper subgroup of G. □

Remark 5. We can formulate Theorem **??** as follows: the map ϕ identifies the homotopy fixed set $(X^\vee)^{hG}$ with the p-profinite completion of the actual fixed set X^G.