SYMPLECTIC GEOMETRY, LECTURE 4

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1. Hamiltonian Vector Fields

Recall from last time that, for (M, ω) a symplectic manifold, $H : M \to \mathbb{R}$ a C^∞ function, there exists a vector field X_H s.t. $i_{X_H}\omega = dH$. Furthermore, the associated flow ρ_t of this vector field is an isotopy of symplectomorphisms.

Example. Consider $S^2 \subset \mathbb{R}^3$ with cylindrical coordinates (r, θ, z) and symplectic form $\omega = d\theta \wedge dz$ (ω is the usual area form). Then setting $H = z$ gives the vector field $\frac{\partial}{\partial\theta}$: the associated flow is precisely rotation by angle t.

Note also that the critical points of H are the fixed points of ρ_t, and ρ_t preserves the level sets of H, i.e.

$$\frac{d}{dt}(H \circ \rho_t) = \frac{d}{dt}(\rho_t^* H) = \rho_t^*(L_{X_H} H) = \rho_t^*(i_{X_H}\omega(X_H)) = \rho_t^*(\omega(X_H, X_H)) = 0 \tag{1}$$

One can apply this to obtain the ordinary formula for conservation of energy.

Definition 1. *X is a* symplectic vector field *if $L_X\omega = 0$, i.e. $i_X\omega$ is closed. X is Hamiltonian if $i_X\omega$ is exact.*

By Poincaré, we see that, locally, symplectic vector fields are Hamiltonian. Globally, we obtain a class $[i_X\omega] \in H^1(M, \mathbb{R})$.

Example. On T^2, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are symplectic vector fields: since dy and dx are not exact, $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are not Hamiltonian.

Now consider time-dependent Hamiltonian functions, i.e. C^∞ maps $\mathbb{R} \times M \to \mathbb{R}, (t, x) \mapsto H_t(x)$. Let $\mathrm{Ham}(M, \omega)$ denote the space of Hamiltonian diffeomorphisms on ω, i.e. the set of diffeomorphisms ρ s.t. $\exists H_t$ with corresponding flow ρ_t satisfying $\rho_1 = \rho$.

Remark. The *Arnold conjecture* states that for M compact, $\phi \in \mathrm{Ham}(M, \omega)$ with nondegenerate $\mathrm{Fix}(\phi)$ (i.e. at a fixed point p, $d\phi(p) - \mathrm{id}$ is invertible),

$$\#\mathrm{Fix}\phi \geq \sum \dim H^i(M) \tag{2}$$

This statement is false for non-Hamiltonian vector fields, as seen in the case of $\frac{\partial}{\partial x}$ on a torus.

We can measure the difference between symplectomorphisms and Hamiltonian symplectomorphisms via the flux function

$$\mathrm{Flux}(\rho_t) = \int_0^1 [i_{X_t}\omega] dt \in H^1(M, \mathbb{R}) \tag{3}$$

In general, the flux depends on the homotopy class of the path from the identity to ρ_1.

Remark. The *Flux conjecture* concerns the integral of the flux on $\pi_1\mathrm{Symp}(M, \omega)$, i.e. the nature of

$$\langle \mathrm{Flux}, \pi_1\mathrm{Symp}(M, \omega)\rangle \subset H^1(M, \mathbb{R}) \tag{4}$$

Geometrically, for $\gamma : S^1 \to M$ a loop, let $\gamma_t = \rho_t \circ \gamma : S^1 \to M$ be the image of γ under ρ and define $\Gamma : [0, 1] \times S^1 \to M$ by $(t, s) \mapsto \gamma_t(s)$.

Problem. $\langle \mathrm{Flux}(\rho_t), [\gamma]\rangle = \mathrm{Area}(\Gamma) = \int_{[0,1]\times S^1} \Gamma^*\omega$.

2. Moser's Theorem

One can ask whether, for a given manifold M, two symplectic structures ω_0, ω_1 are equivalent, i.e. whether there is a symplectomorphism $M \to M$ which pulls back one to the other. In general, $[\omega_0] = [\omega_1]$ does not imply that the two structures are symplectomorphic. To study this question further, we give other notions of equivalence.

Definition 2. *Two forms ω_0, ω_1 are* deformation equivalent *if $\exists (\omega_t)_{t\in[0,1]}$ a continuous family of symplectic forms, and* isotopic *if there is such a family with $[\omega_t]$ constant in $H^2(M, \mathbb{R})$.*

Remark. There exist pairs of symplectic forms with the same cohomology class which are not deformation equivalent, as well as pairs which are deformation equivalent but not isotopic (in dimension ≥ 6).

Let M be a compact manifold with ω_0, ω_1 isotopic symplectic forms (i.e. $\exists \omega_t$ as above with each ω_t nondegenerate).

Theorem 1 (Moser)**.** *$\exists$ an isotopy $\rho_t : M \to M$ s.t. $\rho_t^*\omega_t = \omega_0$.*

That is, (M, ω_0) and (M, ω_1) are symplectomorphic.

Proof. (This technique is known as Moser's trick.) By assumption, $[\omega_t]$ is independent of t, i.e. $[\frac{d\omega_t}{dt}] = 0$. Thus, $\exists \alpha_t$ a 1-form s.t. $\frac{d\omega_t}{dt} = -d\alpha_t$: moreover, we can choose this α_t smoothly w.r.t. to t (via the Poincaré lemma). Since ω_t is nondegenerate, $\exists X_t$ s.t. $i_{X_t}\omega_t = \alpha_t$. Moreover, since M is compact, we have a well-defined flow ρ_t of X_t. Now,

$$\frac{d}{dt}(\rho_t^*\omega_t) = \rho_t^*(L_{X_t}\omega_t) + \rho_t^*\left(\frac{d\omega_t}{dt}\right) = \rho_t^*(di_{X_t}\omega_t + \frac{d\omega_t}{dt}) = 0 \tag{5}$$

Since ρ_0 is the identity, we have our desired isotopy. □

Example. For symplectic forms ω_0, ω_1 with $[\omega_0] = [\omega_1]$, consider the family $\omega_t = t\omega_0 + (1-t)\omega_1$. By the above, if this family is nondegenerate, the two forms are symplectomorphic. In general, there is no reason for this to be true: in dimension 2, it always is. More generally, this follows from compatibility with almost-complex structures.

Theorem 2 (Darboux)**.** *For (M, ω) symplectic, $p \in M$, $\exists U \ni p$ with a coordinate system $(x_1, y_1, \ldots, x_n, y_n)$ s.t. $\omega|_U = \sum dx_i \wedge dy_i$.*

Proof. (T_pM, ω_p) has a standard basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$, so there exist local coordinates $(x_1, y_1, \ldots, x_n, y_n)$ s.t. $\omega_p = \sum dx_i \wedge dy_i$. On a neighborhood U of p, we obtain two symplectic forms: ω and the standard form. The family $\omega_t = (1-t)\omega_0 + t\omega_1$ is one of closed forms: since nondegeneracy is an open condition, we can shrink our neighborhood to assure that ω_t is nondegenerate for each t on some $U' \ni p$. Thus, $\exists \alpha \in \Omega^1(U)$ s.t. $\omega_1 - \omega_0 = -d\alpha$. Subtracting a constant, we can assume $\alpha_p = 0$. Let v_t be the vector field on U s.t. $i_{v_t}\omega_t = \alpha$. Then $\exists U'' \ni p$ s.t. its flow ρ_t is defined $\forall t$. By the Moser's trick, we fnd that $\rho_1^*\omega_1 = \omega_0$, implying that the symplectic form is indeed standard after composing our chosen coordinates with ρ_1. □