8 Lecture 8 (Notes: J. Bernstein)

8.1 Dirac Structures

So far we understand the exact Courant Algebroids

$$0 \to T^* \to E \to T \to 0$$

Which are classified up to isomorphism by $[H] \in H^3(M, \mathbb{R}^3)$ and upon a choice of splitting is isomorphic to $(T \oplus T^*, <,>, [,]_H, \pi : E \to T)$. For $H \in \Omega^3_{cl}$. Always consider (M, E) or (M, H). Geometry in exact Courant Algebroids consists of studying special subbundles $L \subseteq E$.

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Theorem 6. *Suppose that $L \subseteq E$ a subbundle which is closed under $[,]$ (involutive), i.e. $[C^\infty(L), C^\infty(L)] \subseteq C^\infty(L)$. then L must be isotropic or $L = \pi^{-1}(\Delta)$ for $\Delta \subseteq T$ integrable distribution. Note, for $\Delta^k \subseteq T$, $\pi^{-1}(\delta)$ is of dimension $n+k$ and contains T^* (so is not isotropic).*

Proof. Suppose L is involutive, but not isotropic, then there exists $v \in C^\infty(L)$ with $< v, v >_m \neq 0$. Now recall property $[fv, v] = f[v, v] - (\pi(v)f)v + 2 < v, v > df \Rightarrow 2 < v, v > df \in C^\infty(L)$ for all f, as $[fv, v], f[v, v] \in C^\infty(L)$. This implies that $df_m \in L_m$ for all m which tells us that $T^*_m \subseteq L_m$ but T^* is isotropic so $L_m = \pi^{-1}(\Delta_m)$ for $\Delta \neq 0$. Thus $\mathrm{rk} L > n$ evertywhere and so L not isotropic at all points $p \in M$ thus $T^*_p \subseteq L_p$ for all p and so $L = \pi^{-1}(\Delta)$ where Δ is an integrable distribution. □

So interesting involutive subbundles are isotropic subbundles $L \subseteq E$. Recall that the axioms of a Courant Algebroid imply that $[a, a] = \frac{1}{2}\pi^* d < a, a >$. Thus on L, $[,]_{\mathcal{C}}|_{C^\infty(L)}$ defines a Lie Algebroid when L is involutive and isotropic. So $L \subseteq E$ with $[L, L] \subseteq L$ and $< L, L >= 0$ implies that $(L, [,], \pi)$ is a Lie Algebroid which implies $(C^\infty(\wedge^* L^*), d_L)$ gives rise the $H_{d_L}(M)$ the Lie Algebroid Cohomology.

Definition 14. *When an isotropic, involutive $L \subset E$ is maximal it is called a* Dirac Structure

Examples of Dirac structures in $0 \to T^* \to E \to T \to 0$

- $T^* \subset E$ as $[T^*, T^*] \subseteq [T^*, T^*]$
- If we split $(T \oplus T^*, [,]_H)$ then $[X, Y]_H \in C^\infty(T)$ if and only if $H = 0$ so $T \in T \oplus T^*$ is a Dirac structure if and only if $H = 0$
- Any maximal isotropic transverse L (that is such that $L \cap T^* = \{0\}$ is of the form $L = \Gamma_B$. Since $e^B[e^{-B}\cdot, e^{-B}\cdot]_H = [\cdot, \cdot]_{H+dB}$ so $e^B[T, T]_{H-dB} = e^B[e^{-B}\Gamma_B, e^{-B}\Gamma_B]_{H-dB} = [\Gamma_B, \Gamma_B]_H$. Thus $[\Gamma_B, \Gamma_B] \subset \Gamma_B]$ if and only if $[T, T]_{H-dB} \subseteq T$ and this occurs if and only if $H - dB = 0$ so Γ_B is Dirac when and only when $[H] = 0$. In particular when $[H] \neq 0$ there is no Dirac complement to T^*.
- When $\Delta \subset T$ is an integral distribution then $f : \Delta \oplus \mathrm{Ann}\ \Delta \hookrightarrow T \oplus T^*$ is involutive for $[,]_H$ when and only when $f^*H = 0$.
- For $(T \oplus T^*, [,]_H)$ and $\beta \in \wedge^2 T$ we consider Γ_β. This is Dirac if and only if $[\beta, \beta] = -\beta^* H$ where we think of $\beta : T^* \to T$.

Problem. Verify the condition for Γ_β to be Dirac by first showing that $[\xi + \beta(\xi), \eta + \beta(\eta)] = \zeta + \beta(\zeta)$ if and only if $< [\xi + \beta(\xi), \eta + \beta(\eta)], \zeta + \beta(\zeta) >= 0$. And then showing that $< [df + \beta(df), dg + \beta(dg)], dh + \beta(dh) >= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} + H(\beta(df), \beta(dg), \beta(dh)) = (\mathrm{Jac}\{,\} + \beta^* H)(df, dg, dh)$.

Definition 15. *if $[\beta, \beta] = -\beta^* H$ then β is called a* twisted Poisson Structure.

Suppose that β is a twisted Poisson structure, then $e^B\Gamma_\beta$ is not necessarily $\Gamma_{\beta'}$, in particular if β is invertible (as a map $T^* \to T$) and $\beta^{-1} = B$ then $e^{-B}\Gamma_\beta = T$. However if B is "small enough" then $e^B\Gamma_\beta = \Gamma_{\beta'}$. To quantify this we note that $e^B : \xi + \beta(\xi) \mapsto \beta(\xi) + \xi + B\beta(\xi)$ which we want equal to $\eta + \beta'(\eta)$. This happens if and only if $\eta = (1 + B\beta)\xi$ and also $\beta(\xi) = \beta'(\eta) = \beta'(1 + B\beta)\xi$. Thus $\beta' = \beta(1 + B\beta)^{-1}$ and so smallness just means that the map is invertible (i.e. what is written makes sense).

Definition 16. *The transformation from $\beta \mapsto \beta(1 + B\beta)^{-1}$ is called a* gauge transform *of β.*

Problem. (Ševera-Weinstein) Show that if β is Poisson and $d\beta = 0$ then β' is Poisson. Also show that $H^{\cdot}_\beta(M) \cong H^{\cdot}_{\beta'}(M)$, (i.e. one has a isomorphsm of Poisson cohomology. (Hint: $e^B : \Gamma_\beta \to \Gamma_{\beta'}$ is an isomorphism of Lie Algebras).

8.2 Geometry of Lie Groups

Recall that for a Lie group G one has a natural action of $G \times G$ on G, given by $(g,h) \cdot x = gxh = L_g R_h x$ (here one has a left action and a right action). These actions commute in that $(gx)h = g(xh)$. Now for $\mathfrak{g} = T_e G$ the lie algebra of G one has two identifications of $\mathfrak{g} \to T_g G$ namely $a \mapsto a^L|_g = (L_g)_* a$ and $a \mapsto a^R|_g = (R_g)_* a$ where a^L, a^R are left and right invariant vector fields respectively. We have by definition $[a^L, b^L]_{Lie} = [a,b]^L$. Now if $j : G \to G$ is given by $x \mapsto x^{-1}$, then $jL_g = R_{g^{-1}} j$ so $j_*(L_g)_* = (R_{g^{-1}})_* j_*$. In particular since $(j_*)_e = -Id$, one has $(j_* a^L)_{g^{-1}} = j_*(L_g)_* a = (R_{g^{-1}})_* j_* a = -(R_{g^{-1}})_* a = -a^R|_{g^{-1}}$. Thus $j_* a^L = -a^R$. Thus $[a^R, b^R] = [j_* a^L, j_* b^L] = j_*[a^L, b^L] = j_*[a,b]^L = -[a,b]^R$. One also has $[a^L, b^R] = 0$. To see this we note that the map $\mathfrak{g} \to C^\infty(TG)$ given by $a \mapsto a^L|_g = \frac{d}{dt}(g\gamma(t))$ exponentiates to a right action $R_{\gamma(t)}$ similarly a^R exponentiates to a left action and so $[a^L, b^R] = 0$.
We now define $Ad_g : \mathfrak{g} \to \mathfrak{g}$ by $Ad_g(X) = (R_{g^{-1}})_*(L_g)_*$. Equivalently $a^R|_g = (Ad_{g^{-1}} a)^L|_g$. We define $ad_X = d(Ad_g)_0 = [X, \cdot]$.

Lemma 1. *If $\rho \in \Omega^k(G)$ is bi-invariant then $d\rho = 0$*

Proof. If ρ is left invariant then $\rho \in \wedge^k \mathfrak{g}^*$ and so

$$d\rho(X_0, \ldots, X_k) = \sum_i (-1)^i X_i \rho(X_0, \ldots, \hat{X}_i, \ldots, X_k) + \sum_{i,j} (-1)^{i+j} \rho([X_i, X_j], X_0, \ldots, X_k)$$

, where we have chosen $X_0, \ldots X_k$ to be left invariant so the first sum is zero . On the other hand right invariance tells us that for all X, $\sum \rho(X_1, \ldots, [X, X_i], \ldots, X_k) = 0$. $\square$

Problem. Show how the statement above implies that $d\rho = 0$.

We define *Cartan one-forms* to be forms $\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ by $\theta^L_g(v) = (L_{g^{-1}})_* v \in \mathfrak{g}$. and $\theta^R_g(v) = (R_{g^{-1}})_* v \in \mathfrak{g}$. So $\theta^L_x \circ (L_{g^{-1}})_* = \theta^L_{gx}$. Thus θ^L is left invariant as θ^R is right invariant.
For $G = Gl_n$, $\mathfrak{g} = M_n$ one has $\theta^L = g^{-1} dg$ and $\theta^R = dg g^{-1}$. Now if $g = [g_{ij}]$ that is g_{ij} are coordinates one gets matrix of oneforms $[g_{ij}]^{-1}[dg_{ij}]$. Then $(\sigma g)^{-1} d(\sigma g) = g^{-1}\sigma^{-1}\sigma dg = g^{-1} dg$, and so it is left invariant (similarly one can check that the obvious definition is indeed right invariant). At $1 \in GL_n$ one has $\mathfrak{g}$ consisting of $n \times n$ matrices $\{[a_{ij}]\}$ here we make think of $[a_i j] = \sum_{i,j} a_{ij} \frac{\partial}{\partial g_{ij}}$. so $g^{-1} dg (\sum_{i,j} a_{ij} \frac{\partial}{\partial g_{ij}}) = a_i j$, so $g^{-1} dg|_e = Id : \mathfrak{g} \to \mathfrak{g}$. This is also true for θ^L and θ^R.

9 Lecture 9 (Notes: K. Venkatram)

Last time, we talked about the geometry of a connected lie group G. Specifically, for any a in the corresponding Lie algebra $\mathfrak{g}$, one can define $a^L|_g = L_{g*} a$ and choose $\theta^L \in \Omega^1(G, \mathfrak{g})$ s.t. $\theta^L(a^L) = a$. For instance, for GL_n, with coordinates $g = [g_{ij}]$, one has $\theta^L = g^{-1} dg$, and similarly $\theta^R = dg g^{-1}$.This implies that $dg \wedge \theta^L + g d\theta^L = 0 \implies d\theta^L + \theta^L \wedge \theta^L = 0 \implies d\theta^L + \frac{1}{2}[\theta^L, \theta^L] = 0$, the latter of which is the Maurer-Cartan equation.

Problem. 1. Extend this proof so that it works in the general case.

2. Show $j^* \theta^R = -\theta^L$.

3. Show $d\theta^R - \frac{1}{2}[\theta^R, \theta^R] = 0$.

4. Show $\theta^R(a^L)|_g = \mathrm{Ad}\,_g a \forall a \in \mathfrak{g}, g \in G$.