Chapter 6

Curves

6.1 Curves as a map from $\mathbb{R}$ to $\mathbb{R}^n$

As we've seen, we can say that a **parameterized differentiable curve** is a differentiable map α from an open interval $I = (-\epsilon, \epsilon)$ to $\mathbb{R}^n$. Differentiability here means that each of our coordinate functions are differentiable: if α is a map from some I to $\mathbb{R}^3$, and $\alpha = (x(t), y(t), z(t))$, then α being differentiable is saying that $x(t), y(t)$, and $z(t)$ are all differentiable. The vector $\alpha'(t) = (x'(t), y'(t), z'(t))$ is called the tangent vector of α at t.

One thing to note is that, as with our notation for surfaces, our curve is a differentiable map, and not a subset of $\mathbb{R}^n$. do Carmo calls the image set $\alpha(I) \subset \mathbb{R}^3$ the **trace** of α. But multiple "curves," i.e., differentiable maps, can have the same image or trace. For example,

Example 4. *Let*

$$\alpha(t) = (\cos(t), \sin(t)) \tag{6.1}$$
$$\beta(t) = (\cos(2t), \sin(2t)) \tag{6.2}$$
$$\gamma(t) = (\cos(-t), \sin(-t)) \tag{6.3}$$

with t in the interval $(-\epsilon, 2\pi + \epsilon)$.

Then the image of α , β, and γ are all the same, namely, the unit circle centered at the origin. But the velocity vector of β is twice that of α's, and γ runs as fast as α does, but in the opposite direction – the orientation of the curve is reversed. (In general, when we have a curve α defined on (a, b), and define $\gamma(-t) = \alpha(t)$ on $(-b, -a)$, we say α and γ differ by a change of orientation.)

6.2   Arc Length and Curvature

Now we want to describe properties of curves. A natural one to start with is **arc length.** We define the arc length s of a curve α from a time t_0 to a time t as follows

$$s(t) = \int_{t_0}^{t} ds \tag{6.4}$$

$$= \int_{t_0}^{t} \frac{ds}{dt} dt \tag{6.5}$$

$$= \int_{t_0}^{t} |\alpha'(t)| dt \tag{6.6}$$

The next property that we would like to define is **curvature.** We want curvature to reflect properties of the image of our curve – i.e., the actual subset C in $\mathbb{R}^n$ – as opposed to our map. Letting our curvature vector be defined as $\alpha''(t)$ has the problem that, while it measures how fast the tangent vector is changing along the curve, it also measures how large our velocity vector is, which can vary for maps with the same image. Looking back at Example 1, while both α and β's images were the unit circle, the velocity and acceleration vectors were different:

34

$$\alpha'(t) = (-\sin t, \cos t) \tag{6.7}$$

$$\alpha''(t) = (-\cos t, -\sin t), |\alpha''(t)| = 1 \tag{6.8}$$

$$\beta'(t) = (-2\sin(2t), 2\cos(2t)) \tag{6.9}$$

$$\beta''(t) = (-4\cos(2t), -4\sin(2t)), |\beta''(t)| = 4 \tag{6.10}$$

A way to correct this problem is to scale our velocity vector down to unit length – i.e., let $ds/dt = 1$. Then the magnitude of the velocity vector won't skew our curvature vector, and we'll just be able to look at how much the angle between neighboring tangent vectors is changing when we look at $\alpha''(t)$ (how curved our curve is!)

To do this we **parameterize by arc length.** First, we look at the arc length function $s(t) = \int_{t_0}^{t} |\alpha'(t)| dt$. If $\alpha'(t) \neq 0$ for all t, then our function is always increasing and thus has an inverse s^{-1}. So instead of mapping from time, along an interval I, into length, we can map from length into an interval I.

Definition 6.2.1. : *A paramaterized differentiable curve $\alpha : I \to \mathbb{R}^3$ is* **regular** *if $\alpha'(t) \neq 0$ for all $t \in I$.*

Thus to parameterize by arc length, we require our curve to be regular. Then, given a fixed starting point t_0 going to a fixed end point t_1, and the length of our arc from t_0 to t_1 being L, we can reparameterize as follows:

$$(0, L) \to (t_0, t_1) \to \mathbb{R}^3 \text{ (or } \mathbb{R}^n.)$$

where the first arrow is given by s^{-1}, and the second arrow is just our curve α mapping into real-space. If $s \in (0, L)$, and our reparameterized curve is β, then β and α have the same image, and also $|\beta'(s)|$ is $|dt/ds \cdot \alpha'(t)| = 1$. So after reparamterizing by arc length, we have fixed the length of our velocity vector to be 1.

Now we can properly define curvature.

Definition 6.2.2. *Let α be a curve paramaterized by arc length. Then we say that $\alpha''(s)$ (where s denotes length) is the* **curvature vector of** α **at** *s, and the* **curvature at s** *is the norm of the curvature vector, $|\alpha''(s)|$.*

Example 5. *Let's go back to the example of a circle, in this case with radius r.*

$\alpha(t) = (r\sin(t), r\cos(t))$, and $\alpha'(t) = (r\cos t, -r\sin t)$. So $|\alpha'(t)| = r$, and not 1. In order to correct for this, set

$\beta(s) = (r\sin(s/r), r\cos(s/r))$. Then $\beta'(s) = (\cos(s/r), -\sin(s/r))$ and $|\beta'(s)| = 1$. Our circle is now parameterized by arc length.

The curvature vector at a given length s is then

$$\beta''(t) = (-(1/r)\sin(s/r), -(1/r)\cos(s/r)) \tag{6.11}$$

and $|\beta''(s)| = 1/r$. Appropriately, the bigger our circle is, the smaller the curvature.

Exercise 2. *Now we take a catenary, the curve we get if we hang a string from two poles.*

Let $\alpha(t) = (t, \cosh t)$, where $\cosh(t) = (1/2)(e^t + e^{-t})$. Parameterize by arc length and check that it works. The identities $\sinh(t) = (1/2)(e^t - e^{-t})$ and $\sinh^{-1}(t) = \ln(t + (t^2 + 1)^{1/2})$ will be of use.