WHEN IS A MINIMAL SURFACE NOT AREA-MINIMIZING?

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1. Introduction

The “Plateau’s Problem” is the problem of finding a surface with minimal area among all surfaces which have the same prescribed boundary. Let x be a solution to Plateau’s problem for a closed curve Γ and let x^t be a variation of x such that $x_0 = x$ and x^t has the boundary Γ for all t. If $A(t)$ is the area of $x(t)$ bounded by Γ then $A'(0) = 0$ since $t = 0$ is a minimum for A. A surface is *minimal* if it is a critical point of the area function for each closed curve that it spans. A piece of a minimal surface bounded by a closed curve (or a family of closed curves) is called *stable* if the surface is a minimum for the area function associated with that curve. It is called *unstable* if it is a local maximum.

It is readily seen that minimal surfaces don’t necessarily minimize area. Therefore it is natural to ask when a minimal surface is area minimizing. This is in general a difficult problem since it is hard to analyze the area function. Even analyzing the area function and finding its global minimum does not suffice if the boundary is a family of closed curves(see 4.2). In this paper we describe Schwarz’s theorem which gives sufficient conditions for a minimal surface to be unstable thus not area minimizing. Schwarz’s theorem states that if the smallest eigenvalue λ of the laplacian of the Gaussian image of a piece of a minimal surface is greater than two then the piece is unstable. A result due to J. L. Barbosa, M. do Carmo in the converse direction states that if the mentioned eigenvalue is less than two then the piece is stable. Notice however that even these results together leave the case $\lambda = 2$ open.

Section 2 introduces the notation that we use and also provides the background information needed. Section 3 gives an elementary proof to a corollary of Schwarz’s theorem. Section 4 presents a nice application of this corollary and also describes a class of examples to which this corollary does not apply. Section 5 gives an outline of the proof of Schwarz’s theorem. This paper is mainly based on Michael Oprea’s book *The Mathematics of Soap Films: Explorations with Maple, 2000* and the paper *On the Size of a Stable Minimal Surface in* $\mathbb{R}^3$.,J. L. Barbosa, M. do Carmo (1974).

2. Definitions and Background

2.1. **Definitions.** We introduce the fundamental definitions and notations that are used to study Minimal Surfaces Theory. Let U be a nonempty, open, connected and simply connected set in the plane (such a set is called a domain). Let $x : U \to \mathbb{R}^3$ be a regular parametrized surface. Throughout this paper we denote the coefficients of the first fundamental form by

$$E = \langle x_u, x_u \rangle, F = \langle x_u, x_v \rangle, G = \langle x_v, x_v \rangle,$$

the Gauss map by

$$N = \frac{x_u \times x_v}{|x_u \times x_v|},$$

the coefficients of the second fundamental form by

$$e = -\langle N_u, x_u \rangle, f = -\langle N_u, x_v \rangle, g = -\langle N_v, x_v \rangle.$$

Then the Gaussian curvature is

$$K = \frac{eg - f^2}{EG - F^2},$$

and the mean curvature is

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}.$$

x is said to be a minimal surface if $H \equiv 0$, or equivalently,

$$eG - 2fF + gE = 0.$$

A reparametrization of x is a differentiable function $y : V \to \mathbb{R}^3$ so that $y = \phi \circ x$ for some diffeomorphism $\phi : V \to U$. x is said to have isothermal parameters if $E = G$ and $F = 0$. Note that with isothermal parameters, the minimal surface equation reduces to $e + g = 0$ and the Gaussian curvature becomes $K = -\frac{e^2+f^2}{E^2}$.

2.2. **Theorems.** We introduce the Weierstrass Enneper representations of minimal surfaces and explain its relationship to the Gauss map.

Theorem 2.3. *A minimal surface has an isothermal parametrization*

The development of the minimal surface equation and the proof of 2.3 can be found in [Osserman] Let x be a minimal surface with isothermal parameters. It's a well known fact that

$$\Delta x = (2EH)N \qquad (2.3.1)$$

(See [Oprea] or [Osserman]) where Δ and H are the laplacian and the mean curvature of x. Since x is minimal $H = 0$ therefore x is harmonic. Let $x(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$ and define $\phi^i = x^i_u - ix^i_v$ for $i = 1, 2, 3$. ϕ^i is holomorphic since each x^i is harmonic. Let

$\phi = (\phi^1, \phi^2, \phi^3)$ A simple computation shows:

$$(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = E - 2iF - G = 0 \tag{2.3.2}$$

$$|\phi|^2 = |\phi^1|^2 + |\phi^2|^2 + |\phi^3|^2 = 2E \tag{2.3.3}$$

Following theorem is a corrollary of 2.3.1, 2.3.2 and 2.3.3.

Theorem 2.4. *Let $x : U \to \mathbb{R}^3$ be a regular surface. Let $\phi^i = x^i_u - ix^i_v$ and suppose $(\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0$. Then x is minimal if and only if each ϕ^i is holomorphic.*

It is straightforward to check that if ϕ is a holomorphic function satisfying 2.3.2 then

$$x^i = Re[\int \phi^i dz], i = 1, 2, 3$$

defines a minimal surface. So ϕ provides enough information for us to determine the surface. This observation gives us a recipe to construct minimal surfaces: find a holomorphic ϕ satisfying 2.3.2 then integrate to obtain the surface. A nice way to construct such a ϕ is the following:

$$\phi^1 = \tfrac{1}{2} f(1 - g^2),\ \phi^2 = \tfrac{i}{2} f(1 + g^2),\ \phi^3 = fg$$

where f is holomorphic g is meromorphic and fg^2 is holomorphic. On the other hand given ϕ one can set $f = \phi^1 - i\phi^2$ and $g = \frac{\phi^3}{\phi^1 - i\phi^2}$ so that the above equations are satisfied. Therefore we have the following theorem:

Theorem 2.5. ***(Weierstrass Enneper Representation I)****If f is holomorphic, g is meromorphic and fg^2 is holomorphic on a domain U then a minimal surface is defined by (z=u+iv)*

$$x^1(u, v) = Re[\int f(1 - g^2)dz], \tag{2.5.1}$$

$$x^2(u, v) = Re[\int if(1 + g^2)dz], \tag{2.5.2}$$

$$x^3(u, v) = Re[2\int fgdz] \tag{2.5.3}$$

Furthermore if g has an inverse function g^{-1} in a domain D then by making the change of variable $\tau = g$ with $d\tau = g'dz$, and defining $T = f/g'$ we get

Theorem 2.6. ***(Weierstrass Enneper Representation II)*** *For any holomorphic function T, a minimal surface is defined by ($\tau = u + iv$)*

$$x^1(u,v) = Re[\int (1-\tau^2)T(\tau)d\tau], \tag{2.6.1}$$

$$x^2(u,v) = Re[\int i(1+\tau^2)T(\tau)d\tau], \tag{2.6.2}$$

$$x^3(u,v) = Re[2\int \tau T(\tau)d\tau] \tag{2.6.3}$$

The ordered pair (f,g) is called Weierstrass-Enneper data for the surface. A proof of the following can be found in [Oprea].

Theorem 2.7. *(Gaussian curvature in terms of Weierstrass-Enneper data) Given a surface x with Weierstrass-Enneper data (f,g), the Gaussian curvature equals*

$$K = \frac{-4}{|T|^2(1+u^2+v^2)^4}$$

where $T = \frac{f}{g'}$

2.8. **Gauss map and steographic projection.** Let x be a regular surface with isothermal parameters. Since x_u, x_v and N constitute an orthogonal basis one has the following equations for N_u and N_v

$$N_u = \frac{\langle N_u, x_u\rangle}{\langle x_u, x_u\rangle}x_u + \frac{\langle N_u, x_v\rangle}{\langle x_v, x_v\rangle}x_v + \frac{\langle N_u, N\rangle}{\langle N, N\rangle}N = -\frac{e}{E}x_u - \frac{f}{E}x_v \tag{2.8.1}$$

$$N_v = \frac{\langle N_v, x_u\rangle}{\langle x_u, x_u\rangle}x_u + \frac{\langle N_v, x_v\rangle}{\langle x_v, x_v\rangle}x_v + \frac{\langle N_v, N\rangle}{\langle N, N\rangle}N = -\frac{f}{E}x_u - \frac{g}{E}x_v \tag{2.8.2}$$

Now using the above and employing the facts that $e + g = 0$ and $K = -\frac{e^2+f^2}{E^2}$ we get

$$\langle N_u, N_u\rangle = -EK = \langle N_v, N_v\rangle, \langle N_u, N_v\rangle = 0 \tag{2.8.3}$$

Now let $S^2 = \{(x,y,z)|x^2+y^2+z^2 = 1\}$ be the usual 2 dimensional sphere. *Steographic projection from the North pole* N is defined by

$$St : S^2 - N \to \mathbb{R}^2 \tag{2.8.4}$$

$$St(x,y,z) = (\frac{x}{1-z}, \frac{y}{1-z}, 0) \tag{2.8.5}$$

Theorem 2.9. *Let x be a minimal surface with isothermal parameters and Weierstrass-Enneper data (f,g). Then $g = St \circ N$*

Proof A long but straightforward computation yields

$$St(N(u,v)) = \frac{\phi^3}{\phi^1 - i\phi^2}$$

We refer the reader to [Oprea] for details.

3. A Condition for a Minimal Surface to be not Area Minimizing

In this section we show that a surface is not an area minimizer for a piece of the surface whose image under the Gauss map contains more than a hemisphere. (By piece of a surface we mean the image of the surface restricted to a bounded, closed, simply connected subset of the domain)

This theorem is very useful in disproving the area-minimizing of the surfaces. However the converse is not true: there are many examples of minimal surfaces whose Gaussian image does not contain a hemisphere and yet are not area minimizers. We give some of these examples in section 4.

Theorem 3.1. *Let $x : U \to \mathbb{R}^3$ be a regular minimal surface given by Weierstrass-Enneper representation II. Let R be a region in U. If the closed unit disk is contained in the interior of R then $x(R)$ does not have the minimum area among the curves spanning $x(\partial R)$*

Proof Define a family of surfaces y^t by

$$y^t = x + t\rho N$$

where $\rho : R \to \mathbb{R}$ is a C^2 function that vanishes on ∂R. We have

$$y^t_u = x_u + t(\rho_u N + \rho N_u) \tag{3.1.1}$$

$$y^t_v = y_v + t(\rho_v N + \rho N_v) \tag{3.1.2}$$

The coefficients of first fundamental form of y^t are

$$E^t = \langle x_u, x_u\rangle + 2t(\rho_u\langle N, x_u\rangle + \rho\langle N_u, x_u\rangle + t^2(\rho_u^2 + \rho\langle N_u, N_u\rangle) \tag{3.1.3}$$

$$= E + 2t(-\rho e) + t^2(\rho_u^2 - EK) \tag{3.1.4}$$

Similarly

$$F^t = t(-2\rho f) + t^2(\rho_u\rho_v) \tag{3.1.5}$$

$$G^t = E + 2t(\rho e) + t^2(\rho_v^2 - EK) \tag{3.1.6}$$

The determinant of the first fundamental form of y^t is

$$E^tG^t - (F^t)^2 = E^2 + t^2(E(\rho_u^2 + \rho_v^2) - 4\rho^2 e^2 - 4\rho^2 f^2 - 2E^2K) + O(t^3) \tag{3.1.7}$$

$$= E^2 + t^2(E(\rho_u^2 + \rho_v^2) + 2\rho^2 E^2 K) + O(t^3) \tag{3.1.8}$$

The area of $y^t(R)$ is

$$A(t) = \int\int_R \sqrt{E^2 + t^2(E(\rho_u^2 + \rho_v^2) + 2\rho^2 E^2 K) + O(t^3)}dudv$$

Now assume that the integrand has the Taylor series expansion $a_0 + a_1 t + a_2 t^2 + ...$ about zero.Taking the square and equating the coefficients yields

$$a_0 = E, a_1 = 0, a_2 = \frac{1}{2}(\rho_u^2 + \rho_v^2 + 2\rho^2 EK)$$

We therefore obtain

$$A(0) = area(x(R)) \tag{3.1.9}$$

$$A'(0) = 0 \tag{3.1.10}$$

$$A''(0) = \int\int_R (\rho_u^2 + \rho_v^2 + 2\rho^2 EK)dudv \tag{3.1.11}$$

On the other hand by Green's theorem we have

$$\int\int_R (\frac{\partial(\rho\rho_u)}{\partial u} + \frac{\partial(\rho\rho_v)}{\partial v})dudv = \int_{\partial R}(-\rho\rho_u dv + \rho\rho_v du) \tag{3.1.12}$$

Since ρ is zero on the boundary of R we get

$$\int\int_R (\rho_u^2 + \rho_v^2)dudv = -\int\int_R \rho\Delta\rho dudv$$

where $\Delta\rho = \rho_{uu} + \rho_{vv}$ is the Laplacian of ρ. Therefore

$$A''(0) = \int\int_R \rho(2\rho EK - \Delta\rho)dudv \tag{3.1.13}$$

Assume that x minimizes the area $x(R)$ and that R contains a unit disk. Then by 3.1.9 we must have

$$\int\int_R (-\rho\Delta\rho + 2\rho^2 EK)dudv \geq 0$$

for any C^2 function $\rho : U \to \mathbb{R}^2$ that vanishes on the boundary of R.The existence of ρ that makes the integral negative will give us the desired result since then the area of $x(R)$ will be a local maximum for the variation $y^t = x + t\rho N$. If we substitute for $E = 2(1 + u^2 + v^2)^2 |T|^2$ and $K = \frac{-4}{|T|^2(1+u^2+v^2)^4}$ into the integral we obtain

$$A''(0) = \int\int_R (\frac{-8\rho^2}{(1 + u^2 + v^2)^2} - \rho\Delta\rho)dudv \tag{3.1.14}$$

Now let $D(r)$ be the closed disk with center at the origin and radius r.Define

$$\rho^r(u,v) = \frac{u^2+v^2-r^2}{u^2+v^2+r^2}$$

and

$$S(r) = \int\int_{D(r)} \left(\frac{-8(\rho^r)^2}{(1+u^2+v^2)^2} - \rho^r \Delta \rho^r\right) dudv$$

It's easy to check that the integrand becomes zero when $r = 1$. To simplify $S(r)$ let us make a change of variables by taking $u = rs$ and $v = rt$ with $du = rds$ and $dv = rdt$. $\rho^r(u,v) = \rho^1(s,t)$, and by chain rule $\rho^r_u = \rho^1_s/r$ and $\rho^r_{uu} = \rho^1_{ss}/r^2$ similarly for v. Writing ρ instead of ρ^1 for short we have

$$S(r) = -\int\int_{D(1)} \left(\frac{8\rho^2 r^2}{(1+r^2(s^2+t^2))^2} + \rho\Delta\rho\right) dsdt$$

We want to show that $S(r) < 0$ for r slightly larger than 1. So we need to compute $S'(1)$ Since the derivative of the integral equals the integral of the derivative we have

$$S'(r) = -\int\int_{D(1)} \frac{16\rho^2 r(1+r^2(s^2+t^2))^2 - 32\rho^2 r^3(1+r^2(s^2+t^2))(s^2+t^2)}{(1+r^2(s^2+t^2))^4} dsdt$$

At $r = 1$ the expression becomes

$$S'(1) = -\int\int_{D(1)} \frac{16\rho^2(1+s^2+t^2) - 32\rho^2(s^2+t^2)}{(1+s^2+t^2)^3} dsdt \tag{3.1.15}$$

$$= -\int\int_{D(1)} \frac{16\rho^2(1+s^2+t^2-2s^2-2t^2)}{(1+s^2+t^2)^3} dsdt \tag{3.1.16}$$

$$= 16\int\int_{D(1)} \frac{(s^2+t^2-1)^3}{(s^2+t^2+1)^3} dsdt \tag{3.1.17}$$

The integral is negative because the numerator of the integrand is always negative. We know that $S(1) = 0$ and $S'(1) < 0$ therefore there exists $\overline{r}$ so that $S(r) < 0$ whenever $1 < r < \overline{r}$. Now fix $1 < r < \overline{r}$ and define $\tilde{\rho} : R \to \mathbb{R}$:

$$\tilde{\rho}(u,v) = \begin{cases} \rho^r(u,v) & \text{if } (u,v) \text{ is in } D(r) \\ 0 & \text{otherwise} \end{cases}$$

Now for $\rho = \tilde{\rho}$ the right hand side of (3.1.14) is equal to $S(r)$ and thus is negative. Note that the derivatives of $\tilde{\rho}$ have discontinuities on $D(r)$ but we can round them off keeping $A''(0)$ negative.

Finally note that the domain of T in Weierstrass-Enneper representation is the same as the image of the Gauss map composed with Steographic projection from the North pole. Therefore a minimal surface given in Weierstrass Enneper II parametrization contains the unit disk if and only if the Gauss map contains the Southern hemisphere. On the other hand by a change of variables we can change any point on the image of the Gauss map into the North pole. Therefore we deduce

Theorem 3.2. *A piece of a minimal surface is not area minimized by the surface if its image under the Gauss map contains a hemisphere.*

4. Examples and Non-examples

4.1. **Enneper's Surface.** A nice application of 3.1 is to take a closed curve in $\mathbb{R}^3$ together with two minimal surfaces spanning the curve, one of which is the actual area minimizer and the other of whose Gauss map contains a hemisphere, and compare the areas enclosed by the surfaces. However this turns out to be very difficult because there seems to be no example of two or more explicit minimal surfaces spanning a given curve(see [Oprea] p. 114). Now consider Enneper's surface $x(u,v) = (u - u^3/3 + uv^2, -v + v^3/3 - vu^2, u^2 - v^2)$. It is known that this curve has no self intersections for $u^2 + v^2 < 3$ and it is not hard to see that $g(z) = z$. Therefore the Gaussian image of any domain U that contains the unit disk and is contained in $(u,v)|u^2 + v^2 < 3$ will contain the Southern hemisphere and hence is not the area minimizer for the curve C determined by applying Enneper's surface's parametrization to the boundary of U. But by Douglas and Radó we know that there is a solution to Plateau's problem for the curve C. So there exists two minimal surfaces spanning C.

4.2. **Catenoid.** Now consider the catenoid. It is known that the catenoid is the solution to Plateau's problem for two identical and parallel circles. On the other hand one might begin with a catenoid and compare the areas of two disks with the area of the part of the catenoid bounded by the circles. A Weierstrass-Enneper II representation of the catenoid is given by $T(\tau) = \frac{1}{2\tau^2}$. In order to simplify integrals choose $\tau = -e^z$. For this choice of τ Weierstrass-Enneper data of catenoid is $(f,g) = (-\frac{e^z}{2}, -e^z)$ Note that g never takes the value 0. Therefore the domain of τ can never contain the unit disk. Letting $z = u + iv$ gives

$$x(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$$

Easy computation yields:

$$E = \cosh^2 u = G, F = 0$$

Now note that the image of the curve $u = c$ corresponds to a disk with radius $\cosh c$ on the catenoid and a disk with radius e^c on τ plane. Therefore the part of the catenoid bounded by the circles with equal radii correspond to the infinite strip between $u = c$ and $u = -c$ on (u,v) plane and the annulus bounded by the circles centered at origin of radii e^c and e^{-c} on

τ plane. Simple calculus gives

$$\text{the lateral area} = 2\pi \int_{-c}^{c} \cosh^2 u du = \frac{\pi(e^{2c} - e^{-2c} + 8c)}{2}$$

the total area of two disks equals

$$2\pi \cosh^2 c = \frac{\pi(e^{2c} + e^{-2c} + 2)}{2}$$

Plotting by matlab shows that the total area of the disks become smaller than the lateral area if $c > 0.64$. We finish this section by stating the conclusions that we draw from the above discussion:

Given a surface with Weierstrass-Enneper data (f, g) with g holomorphic , theorem 3.1 does not apply if the image of g does not contain the origin. It also does not apply if the image of the Gauss map skips two antipodal points on the sphere.

5. Schwarz's Theorem

We first introduce the notation and the background needed for Schwarz's theorem and then we give an outline of its proof. Then we show that theorem 3.1 is a corollary of Schwarz's theorem.

Let M be a two-dimensional, orientable compact C^∞ manifold. A domain $D \subset M$ is an open, connected subset with compact closure $\overline{D}$ and such that the boundary ∂D is a finite union of piecewise smooth curves. Let $x : M \to \mathbb{R}^3$ be a minimal surface into the Euclidean space $\mathbb{R}^3$.(here we require x to be only piecewise smooth not necessarily smooth) By the construction of minimal surface equation we know that D is a critical point of the area for all variations of $\overline{D}$ which keep ∂D fixed. When this critical value is a minimum for all such variations, we say that D is *stable*.

Now we recall the formula for the laplacian operator on 2-manifolds in terms of the parametrization (u_1, u_2):

$$\Delta_M = \frac{1}{detG^{1/2}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u^i}(detG^{1/2} g^{ij} \frac{\partial}{u^j}) \tag{5.0.1}$$

Where G is the matrix for the first fundamental form and $(g^{ij}) = G^{-1}$. If we view a surface with isothermal coordinates as a manifold 5.0.1 gives

$$\Delta_M = \frac{1}{E_M}\Delta \tag{5.0.2}$$

Where we have used E_M to denote the E-coefficient of the first fundamental form of the manifold M. Therefore (3.1.13) becomes

$$A''(0) = \int_R \rho(-\Delta_M \rho + 2\rho K) dM \tag{5.0.3}$$

Where $dM = Edudv$ is the area element of M. If $A''(0) > 0$ for all ρ (here we again assume that ρ is only piecewise smooth) then D is stable. We say that D is *unstable* if for some ρ, $A''(0) < 0$.

We denote by $H(D)$ the space of C^∞ functions on $\overline{D}$ which are not identically zero and vanish on ∂D. We call a positive real number λ an eigenvalue in D for Δ_M if there exists $u \in H(D)$ such that $\Delta_M u + \lambda u = 0$. The eigenspace associated with λ is

$$P_\lambda(D) = \{u \in H(D) | \Delta_M u + \lambda u = 0\}$$

It is known that the eigenvalues form a discrete set of positive numbers. We denote the smallest of them by λ_1

Theorem 5.1. ***(Schwarz's Theorem*** [BdC]***)*** *Let $D \subset M$ be a domain and assume that the Gauss map N, restricted to $\overline{D}$ is a branched covering onto $N(\overline{D})$. Assume further that the first eigenvalue for the laplacian Δ_{S^2} in $N(\overline{D})$ is smaller than two. Then D is unstable.*

Proof. Let λ_1 be the first eigenvalue for Δ_{S^2} in $N(\overline{D})$. Then there is a function $u : S^2 \to \mathbb{R}$ satisfying $\Delta_{S^2} u + \lambda_1 u = 0$ Note that N is a parametrization of $N(\overline{D})$. Furthermore by (2.8.3) we know that this parametrization is isothermal. By (2.8.3) and 5.0.2 we have

$$\Delta_{S^2} = -\frac{1}{EK}\Delta = -\frac{1}{K}\Delta_M.$$

Now write u in terms of this new parametrization by letting $f = u \circ N$.(Note here that f is an eigenfunction of the laplace operator Δ_{S^2} with eigenvalue λ_1, since the laplace operator does not depend on the parametrization.) Then by the above equations we have

$$\Delta_M f = \lambda_1 K f$$

If we compute the second derivative of the area for the normal variation fN we obtain

$$A''(0) = \int_{\overline{D}} f(-\Delta_M f + 2fK) dM = \int_{\overline{D}} f^2(-K)(\lambda_1 - 2) dM$$

Since the integrand is always negative we deduce $A''(0) < 0$ hence D is not stable.

□

Following lemma relates the first eigenvalues of two domains one contained in the other. We refer the reader to [BdC] for its proof:

Lemma 5.2. *If $\tilde{D} \subset D$ then $\tilde{\lambda_1} \geq \lambda_1$, and equality holds if and only if $\tilde{D} = D$. ($\tilde{\lambda_1}, \lambda_1$ are the first eigenvalues of $\tilde{D}$ and D)*

5.2 makes sense since if $u \in P_{\lambda_1}(\tilde{D})$ then one can extend u to a function in $H(D)$ by letting u vanish outside $\tilde{D}$. Therefore λ_1 is an eigenvalue of the Laplace operator in $\overline{D}$. So the smallest eigenvalue of D is less than or equal to that of $\tilde{D}$ (one has to make a more careful analysis to show that they are not equal). We show that 3.1 is a corollary of Schwarz's theorem.

proof of 3.1 Let $u : S^2 \to \mathbb{R}$ be the restriction of the projection $u(x, y, z) = z$ to S^2. It is easy to verify that $\Delta_{S^2} u + 2u = 0$. Furthermore $u = 0$ on the great circle determined by $z = 0$. So the hemisphere has eigenvalue 2. By 5.2 any region that contains the hemisphere has an eigenvalue less than two thus by 5.1, it is not stable. Q.E.D.

Note here that if our surface is given by Weierstrass-Enneper Representation II then the composition of projection to third coordinate and the Gauss map is $\frac{u^2+v^2-1}{u^2+v^2+1}$ which explains the mystery behind the choice of the function ρ in the proof of 3.1.

There are other domains on the sphere that have 2 as the first eigenvalue. It is easily checked that the restriction to S^2 of

$$u_{c,z}(x, y, z) = 2 - z \log \frac{(1+z)}{(1-z)} + cz, z \neq 1, 0 \leq c < \infty$$

is a solution to $\Delta_{S^2} u + 2u = 0$. It is easy to see that there are two zeros z_1, z_2 of u as a function of z. If C is the ring shaped domain on the sphere bounded by the circles $z = z_1$ and $z = z_2$ then Schwarz's theorem can be applied using C instead of hemisphere. One can also construct other domains with first eigenvalue two by considering the linear combinations of $u_{c,x}, u_{c,y}, u_{c,z}$. We finish the paper by quoting the following companion of Scwarz's theorem from [BdC]:

Theorem 5.3. ***(do Carmo and Barbosa)*** *Let $D \in M$ be a domain and assume that the first eigenvalue λ_1 of $N(D)$ is greater than two. Then D is stable.*

References

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