18.S34 (FALL 2007)
LIMIT PROBLEMS

1. Let a and b be positive real numbers. Prove that

$$\lim_{n\to\infty} (a^n + b^n)^{1/n}$$

equals the larger of a and b. What happens when $a = b$?

2. Show that $\lim_{n\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log(n)\right)$ exists and lies between $\frac{1}{2}$ and 1.

Note. This number, known as *Euler's constant* and denoted γ, is probably the third most important constant in the theory of complex variables, after π and e. Numerically we have

$$\gamma = 0.5772156649015328606065120900824024310421593359399 2\cdots.$$

It is a famous unsolved problem to decide whether γ is irrational.

3. (47P) If (a_n) is a sequence of numbers such that, for $n \geq 1$,

$$(2 - a_n)a_{n+1} = 1,$$

prove that $\lim_{n\to\infty} a_n$ exists and equals 1.

4. Let K be a positive real number. Take an arbitrary positive real number x_0 and form the sequence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{K}{x_n}\right).$$

Show that $\lim_{n\to\infty} x_n = \sqrt{K}$. (Remark. this is how most calculators determine $\sqrt{K}$.)

5. (70P) Given a sequence (x_n) such that $\lim_{n\to\infty} (x_n - x_{n-2}) = 0$, prove that

$$\lim_{n\to\infty} \frac{x_n - x_{n-1}}{n} = 0.$$

6. Let $x_{n+1} = x_n^2 - 6x_n + 10$. For what values of x_0 is $\{x_n\}$ convergent, and how does the value of the limit depend on x_0?

7. (90P) Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n, m = 0, 1, 2, \ldots)$? Justify your answer.

8. Let $x_0 = 1$ and $x_{n+1} = x_n + 10^{-10^{x_n}}$. Does $\lim_{n\to\infty} x_n$ exist? Explain.

9. (00P) Show that the improper integral

$$\lim_{B\to\infty} \int_0^B \sin(x)\,\sin(x^2)\,dx$$

converges.

10. Let $x > 0$. Define $a_1 = x$ and $a_{n+1} = x^{a_n}$ for $n \geq 1$. For which x does $\lim_{n\to\infty} a_n$ exist (and is finite)?

PART II

LIMITS. Two useful techniques are:

(a) *L'Hôpital's rule.* If $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, then

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)},$$

provided the derivatives in question exist. Some limits can be converted to this form by first taking logarithms, or by substituting $1/x$ for x, etc.

(b) If $f(x)$ is reasonably well-behaved (e.g., continuous) on the closed interval $[a, b]$, then

$$\lim \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \int_a^b f(x)dx,$$

where the limit is over any sequence of "partitions of $[a, b]$" $a = x_0 < x_1 < \cdots < x_n = b$ such that the maximum value of $x_i - x_{i-1}$ approaches 0. In particular, taking $a = 0$, $b = 1$, $x_i = i/n$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(i/n) = \int_0^1 f(x)dx.$$

Sometimes a limit of products can be converted to this form by taking logarithms.

The next problems are all from the Putnam Exam.

11. Let $a > 0$, $a \neq 1$. Find

$$\lim_{x\to\infty} \left(\frac{1}{x} \frac{a^x - 1}{a - 1} \right)^{1/x}$$

12. Find

$$\lim_{n\to\infty} \left[\frac{1}{n^4} \prod_{i=1}^{2n} (n^2 + i^2)^{1/n} \right]$$

13. Let $0 < a < b$. Evaluate

$$\lim_{t\to 0} \left[\int_0^1 (bx + a(1-x))^t dx \right]^{1/t}$$

14. Evaluate

$$\lim_{x\to 0} \frac{1}{x} \int_0^x (1 + \sin(2t))^{1/t} dt$$

15. Evaluate

$$\lim_{n\to\infty} \sum_{j=1}^{n^2} \frac{n}{n^2 + j^2}$$

16. Evaluate

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$

17. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\left\lfloor \frac{2n}{k} \right\rfloor - 2 \left\lfloor \frac{n}{k} \right\rfloor \right).$$

Express your answer in the form $\log(a) - b$, where a and b are positive integers.

18. Evaluate

$$\sqrt[8]{2207 - \frac{1}{2207 - \frac{1}{2207 - \cdots}}}.$$

Express your answer in the form $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are integers.

19. Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim a_n/n = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

20. Evaluate
$$\lim_{x\to 1^-} \prod_{n=0}^{\infty} \left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

21. Let k be an integer greater than 1. Suppose $a_0 > 0$, and define
$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$
for $n > 0$. Evaluate
$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$