18.S66 PROBLEMS #4

Spring 2003

A *tree* T on $[n]$ is a graph with vertex set $[n]$ which is connected and contains no cycles. Equivalently, as is easy to see, T is connected and has $n-1$ edges. A *forest* is a graph for which every connected component is a tree. A *rooted tree* is a tree with a distinguished vertex u, called the *root*. If there are $t(n)$ trees on $[n]$ and $r(n)$ rooted trees, then $r(n) = nt(n)$ since there are n choices for the root u. A *planted forest* (sometimes called a *rooted forest*) is a graph for which every connected component is a rooted tree.

106. [2.5] The number of trees $t(n)$ on $[n]$ is $t(n) = n^{n-2}$. Hence the number of rooted trees is $r(n) = n^{n-1}$.

107. [1] The number of planted forests on $[n]$ is $(n+1)^{n-1}$.

108. [2] Let $S \subseteq [n]$, $\#S = k$. The number $p_S(n)$ of planted forests on $[n]$ whose root set is S is given by

$$p_S(n) = kn^{n-k-1}.$$

109. [2] Given a planted forest F on $[n]$, let $\deg(i)$ be the *degree* (number of children of i). E.g., $\deg(i) = 0$ if and only if i is a leaf (endpoint) of F. If F has k components then it is easy to see that $\sum_i \deg(i) = n - k$. Given $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{N}^n$ with $\sum \delta_i = n - k$, let $N(\delta)$ denote the number of planted forests F on $[n]$ (necessarily with k components) such that $\deg(i) = \delta_i$ for $1 \leq i \leq n$. Then

$$N(\delta) = \binom{n-1}{k-1}\binom{n-k}{\delta_1, \ldots, \delta_n},$$

where $\binom{n-k}{\delta_1,\ldots,\delta_n}$ denotes a multinomial coefficient.

110. [2] The number of trees with $n+1$ unlabelled vertices and n labelled *edges* is $(n+1)^{n-2}$.

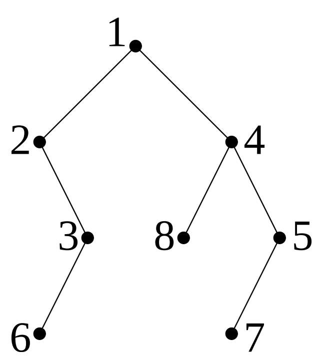
111. [2.5] A *k-edge colored tree* is a tree whose edges are colored from a set of k colors such that any two edges with a common vertex have

different colors. Show that the number $T_k(n)$ of k-edge colored trees on the vertex set $[n]$ is given by

$$T_k(n) = k(nk-n)(nk-n-1)\cdots(nk-2n+3) = k(n-2)!\binom{nk-n}{n-2}.$$

(This problem has received little attention and may be easy.)

112. A *binary tree* is a rooted tree such that every vertex v has exactly two subtrees L_v, R_v, possibly empty, and the set $\{L_v, R_v\}$ is linearly ordered, say as (L_v, R_v). We call L_v the *left subtree* of v and draw it to the left of v. Similarly R_v is called the *right subtree* of v, etc. A binary tree on the vertex set $[n]$ is *increasing* if each vertex is smaller that its children. An example of such a tree is given by:



(a) [1] The number of increasing binary trees on $[n]$ is $n!$.

(b) [2] The number of increasing binary trees on $[n]$ for which exactly k vertices have a left child is the Eulerian number $A(n, k+1)$.

113. An *increasing forest* is a planted forest on $[n]$ such that every vertex is smaller than its children.

(a) [1] The number of increasing forests on $[n]$ is $n!$.

(b) [2] The number of increasing forests on $[n]$ with exactly k components is equal to the number of permutations $w \in \mathfrak{S}_n$ with k cycles.

(c) [2] The number of increasing forests on $[n]$ with exactly k endpoints is the Eulerian number $A(n, k)$.

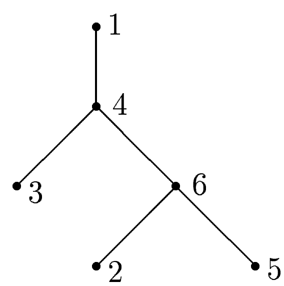
114. [2] Show that

$$\sum_{n\geq 0}(n+1)^n\frac{x^n}{n!} = \left(\sum_{n\geq 0} n^n\frac{x^n}{n!}\right)\left(\sum_{n\geq 0}(n+1)^{n-1}\frac{x^n}{n!}\right).$$

115. [2] Show that

$$\frac{1}{1-\sum_{n\geq 1} n^{n-1}\frac{x^n}{n!}} = \sum_{n\geq 0} n^n\frac{x^n}{n!}.$$

116. [3] Let τ be a rooted tree with vertex set $[n]$ and root 1. An *inversion* of τ is a pair (i,j) such that $1 < i < j$ and the unique path in τ from 1 to i passes through j. For instance, the tree below has the inversions $(3,4), (2,4), (2,6)$, and $(5,6)$.



Let $\mathrm{inv}(\tau)$ denote the number of inversions of τ. Define

$$I_n(t) = \sum_{\tau} t^{\mathrm{inv}(\tau)},$$

summed over all n^{n-2} trees on $[n]$ with root 1. For instance,

$$\begin{aligned}
I_1(t) &= 1\\
I_2(t) &= 1\\
I_3(t) &= 2+t\\
I_4(t) &= 6+6t+3t^2+t^3\\
I_5(t) &= 24+36t+30t^2+20t^3+10t^4+4t^5+t^6\\
I_6(t) &= 120+240t+270t^2+240t^3+180t^4+120t^5+70t^6+35t^7\\
&\quad +15t^8+5t^9+t^{10}.
\end{aligned}$$

Show that

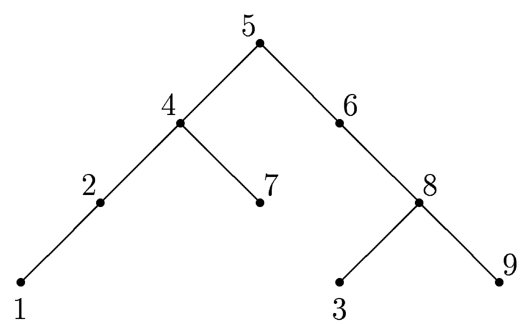
$$t^{n-1}I_n(1+t) = \sum_G t^{e(G)},$$

summed over all *connected* graphs G (without loops or multiple edges) on the vertex set $[n]$, where $e(G)$ is the number of edges of G.

117. (*) An *alternating tree* on $[n+1]$ is a tree with vertex set $[n+1]$ such that every vertex is either less than all its neighbors or greater than all its neighbors. Let $f(n)$ denote the number of alternating trees on $[n+1]$, so $f(1)=1$, $f(2)=2$, $f(3)=7$, $f(4)=36$, etc. Then

$$f(n) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(k+1)^{n-1}.$$

118. [2.5] A *local binary search tree* is a binary tree, say with vertex set $[n]$, such that the left child of a vertex is smaller than its parent, and the right child of a vertex is larger than its parent. An example of such a tree is:



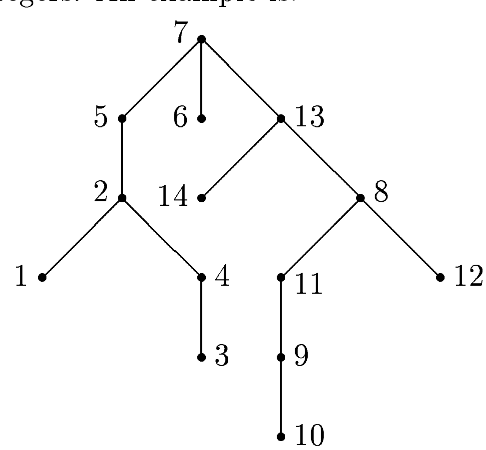
The number $f(n)$ of alternating trees on $[n]$ is equal to the number of local binary search trees on $[n+1]$.

119. (*) A *tournament* is a directed graph with no loops (edges from a vertex to itself) and with exactly one edge $u \to v$ or $v \to u$ between any two distinct vertices u, v. Thus the number of tournaments on $[n]$ (i.e., with vertex set $[n]$) is $2^{\binom{n}{2}}$. Write $C = (c_1, c_2, \ldots, c_k)$ for the directed cycle with edges $c_1 \to c_2 \to \cdots \to c_k \to c_1$ in a tournament on $[n]$. Let $\operatorname{asc}(C)$ be the number of integers $1 \le i \le k$ for which $c_{i-1} < c_i$, and let $\operatorname{des}(C)$ be the number of integers $1 \le i \le k$ for

which $c_{i-1} > c_i$, where by convention $c_0 = c_k$. We say that the cycle C is *ascending* if $\text{asc}(C) \geq \text{des}(C)$. For example, the cycles (a,b,c), (a,c,b,d), (a,b,d,c), and (a,c,d,b) are ascending, where $a < b < c < d$. A tournament T on $[n]$ is *semiacyclic* if it contains no ascending cycles, i.e, if for any directed cycle C in T we have $\text{asc}(C) < \text{des}(C)$. The number of semiacyclic tournaments on $[n]$ is equal to the number of alternating trees on $[n]$. (This problem, usually stated in a different but equivalent form, has received a lot of attention. A solution would be well worth publishing.)

120. [2] An *edge-labelled alternating tree* is a tree, say with $n+1$ vertices, whose edges are labelled $1, 2, \ldots, n$ such that no path contains three consecutive edges whose labels are increasing. (The vertices are not labelled.) If $n > 1$, then the number of such trees is $n!/2$.

121. [2.5] A *recursively labelled tree* is a rooted tree on the vertex set $[n]$, such every subtree (i.e., every vertex and its descendants) consists of consecutive integers. An example is:



Similarly define a *recursively labelled forest.* Let t_n (respectively, f_n) denote the number of recursively labelled trees (respectively, forests) on the vertex set $[n]$. Then t_n is the number of ordered pairs of ternary trees with a total of $n-1$ vertices. (A *ternary tree* is a rooted unlabelled tree such that every vertex has three subtrees, which may be empty, and these subtrees are linearly ordered.) Similary f_n is the number of ternary trees with n vertices.

NOTE. It is known that

$$t_n = \frac{1}{n}\binom{3n-2}{n-1}, \quad f_n = \frac{1}{2n+1}\binom{3n}{n},$$

though these formulas are not relevant to finding a bijective proof.

122. [2] A tree on a linearly ordered vertex set is called *noncrossing* if ik and jl are not both edges whenever $i < j < k < l$. The number of noncrossing trees on $[n]$ is equal to the number of ternary trees with $n-1$ vertices.

123. [2] A *spanning tree* of a graph G is a subgraph of G which is a tree and which uses every vertex of G. The number of spanning trees of G is denoted $c(G)$ and is called the *complexity* of G. Thus Problem 106 is equivalent to the statement that $c(K_n) = n^{n-2}$, where K_n is the complete graph on n vertices (one edge between every two distinct vertices). The *complete bipartite graph* K_{mn} has vertex set $A \cup B$, where $\#A = m$ and $\#B = n$, with an edge between every vertex of A and every vertex of B (so mn edges in all). Then $c(K_{mn}) = m^{n-1}n^{m-1}$.

124. (*) The *n-cube* C_n (as a graph) is the graph with vertex set $\{0,1\}^n$ (i.e., all binary n-tuples), with an edge between u and v if they differ in exactly one coordinate. Thus C_n has 2^n vertices and $n2^{n-1}$ edges. Then

$$c(C_n) = 2^{2^n-n-1}\prod_{k=1}^{n} k^{\binom{n}{k}}.$$

125. [2.5] A *parking function* of length n is a sequence $(a_1, \ldots, a_n) \in \mathbb{P}^n$ such that its increasing rearrangement $b_1 \le b_2 \le \cdots \le b_n$ satisfies $b_i \le i$. The parking functions of length three are 111, 112, 121, 211, 122, 212, 221, 113, 131, 311, 123, 132, 213, 231, 312, 321. The number of parking functions of length n is $(n+1)^{n-1}$.

126. [3] Let $\mathrm{PF}(n)$ denote the set of parking functions of length n. Then

$$\sum_{(a_1,\ldots,a_n)\in \mathrm{PF}(n)} q^{a_1+\cdots+a_n} = \sum_{\tau} q^{\binom{n+1}{2}-\mathrm{inv}(\tau)},$$

where τ ranges over trees on $[n+1]$ with root 1, and where $\mathrm{inv}(\tau)$ is defined in Problem 116.

127. [2.5] A *valid* n-pair consists of a permutation $w = a_1 \cdots a_n \in \mathfrak{S}_n$, together with a collection I of pairs (i, j) such that

- If $(i,j) \in I$ then $1 \leq i < j \leq n$.
- If $(i,j) \in I$ then $a_i < a_j$.
- If $(i,j),(i',j') \in I$ and $\{i, i+1, \ldots, j\} \subseteq \{i', i'+1, \ldots, j')$, then $(i,j) = (i',j')$.

For example, let $n = 3$. For each $w \in \mathfrak{S}_3$ we put after it the number of sets I for which (w, I) is a valid 3-pair: 123 (5), 213 (3), 132 (3), 231 (2), 312 (2), 321 (1). The number of valid n-pairs is $(n+1)^{n-1}$.

128. (a) [3] Let T be a tournament on $[n]$, as defined in Problem 119. The *outdegree* of vertex i, denoted outdeg(i), is the number of edges pointing out of i, i.e., edges of the form $i \to j$. The *outdegree sequence* of T is defined by

$$\mathrm{out}(T) = (\mathrm{outdeg}(1), \ldots, \mathrm{outdeg}(n)).$$

For instance, there are eight tournaments on [3], but two have outdegree sequence $(1,1,1)$. The other six have distinct outdegree sequences, so the total number of distinct outdegree sequences of tournaments on [3] is 7. The total number of distinct outdegree sequences of tournaments on $[n]$ is equal to the number of forests on $[n]$.

(b) [3] More generally, let G be an (undirected) graph on $[n]$. An *orientation* $\mathfrak{o}$ of G is an assignment of a direction $u \to v$ or $v \to u$ to each edge uv of G. The *outdegree sequence* of $\mathfrak{o}$ is defined analogously to that of tournaments. The number of distinct outdegree sequences of orientations of G is equal to the number of spanning forests of G.

129. (*) Let G be a graph on $[n]$. The *degree* of vertex i, denoted deg(i), is the number of edges incident to i. The (ordered) *degree sequence* of G is the sequence $(\deg(1), \ldots, \deg(n))$. The number $f(n)$ of distinct degree sequences of simple (i.e., no loops or multiple edges) graphs on $[n]$ is given by

$$f(n) = \sum_Q \max\{1, 2^{d(Q)-1}\},$$

where Q ranges over all graphs on $[n]$ for which every connected component is either a tree or has exactly one cycle, which is of odd length. Moreover, $d(Q)$ denotes the number of (odd) cycles in Q.

130. [3] The number of ways to write the cycle $(1, 2, \ldots, n) \in \mathfrak{S}_n$ as a product of $n-1$ transpositions (the minimum possible) is n^{n-2}. (A *transposition* is a permutation $w \in \mathfrak{S}_n$ with one cycle of length two and $n-2$ fixed points.) For instance, the three ways to write $(1, 2, 3)$ are (multiplying right-to-left) $(1, 2)(2, 3)$, $(2, 3)(1, 3)$, and $(1, 3)(1, 2)$.

    NOTE. It is not difficult to show bijectively that the number of ways to write *some* n-cycle as a product of $n-1$ transpositions is $(n-1)!\, n^{n-2}$, from which the above result follows by "symmetry." However, a direct bijection between factorizations of a *fixed* n-cycle such as $(1, 2, \ldots, n)$ and labelled trees (say) is considerably more difficult.

131. [3.5] Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of n with $\lambda_\ell > 0$, and let w be a permutation of $1, 2, \ldots, n$ whose cycles have lengths $\lambda_1, \ldots, \lambda_\ell$. Let $f(\lambda)$ be the number of ways to write $w = t_1 t_2 \cdots t_k$ where the t_i's are transpositions that generate all of $\mathfrak{S}_n$, and where k is minimal with respect to the condition on the t_i's. (It is not hard to see that $k = n + \ell - 2$.) Show that

$$f(\lambda) = (n + \ell - 2)!\, n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i+1}}{\lambda_i!}.$$

    NOTE. Suppose that $t_i = (a_i, b_i)$. Let G be the graph on $[n]$ with edges $a_i b_i$, $1 \le i \le k$. Then the statement that the t_i's generate $\mathfrak{S}_n$ is equivalent to the statement that G is connected.