18.S66 PROBLEMS #5

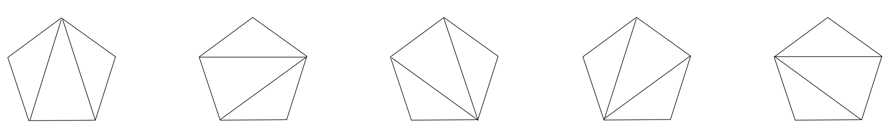
Spring 2003

Let us define the nth *Catalan number* C_n by

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0. \tag{5}$$

Thus $(C_0, C_1, \ldots) = (1, 1, 2, 5, 14, 42, 132, 429, \ldots)$. There are a huge number of combinatorial interpretations of these numbers; 66 appear in Exercise 6.19 of R. Stanley, *Enumerative Combinatorics*, vol. 2. This exercise (as well as some related ones) is available at `www-math.mit.edu/~rstan/ec`, and an addendum with many more interpretations may be found at the same website. We give here a subset of these interpretations that are the most fundamental or most interesting. Problem 143 is perhaps the easiest one to show bijectively is counted by (5). All your other proofs should be bijections with previously shown "Catalan sets." Each interpretation is illustrated by the case $n = 3$, which hopefully will make any undefined terms clear. Needless to say, you should not hand in a problem whose solution you have obtained from an outside source (except reasonable collaboration with other students in the course).

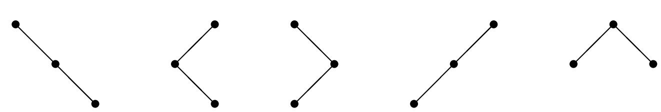
132. [1.5] triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors



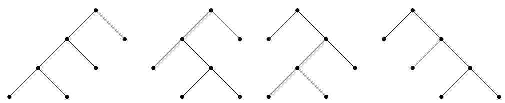
133. [1.5] binary parenthesizations of a string of $n+1$ letters

$$(xx \cdot x)x \quad x(xx \cdot x) \quad (x \cdot xx)x \quad x(x \cdot xx) \quad xx \cdot xx$$

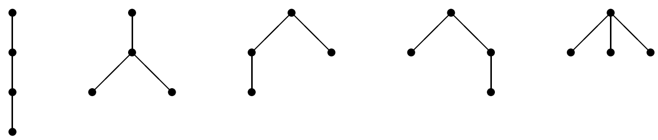
134. binary trees with n vertices



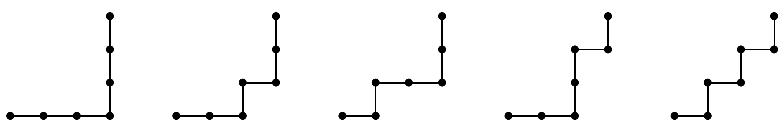
135. [1.5] plane binary trees with $2n+1$ vertices (or $n+1$ endpoints) (A *plane binary tree* is a binary tree for which every vertex is either an endpoint or has two children.)



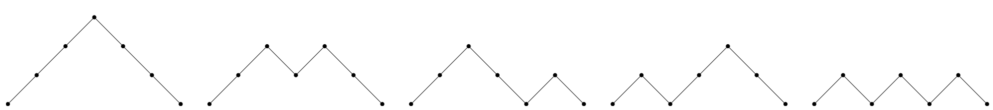
136. [2] plane trees with $n+1$ vertices (A *plane tree* is a rooted tree for which the subtrees of every vertex are linearly ordered from left to right.)



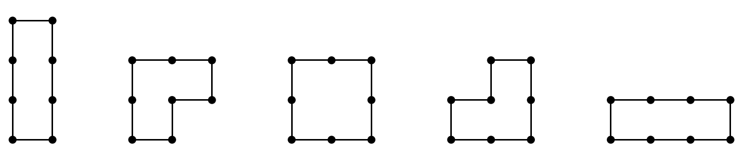
137. [1.5] lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y=x$



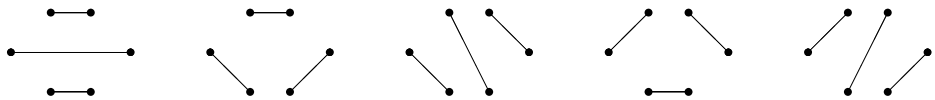
138. [1] Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



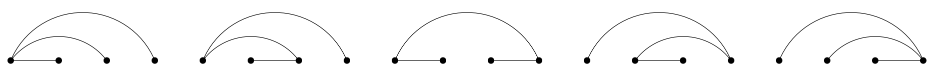
139. [2.5] (unordered) pairs of lattice paths with $n+1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, and only intersecting at the beginning and end



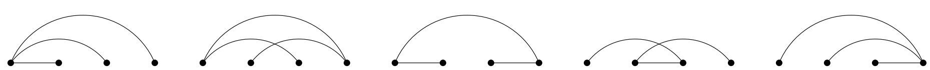
140. [1.5] n nonintersecting chords joining $2n$ points on the circumference of a circle



141. [2] ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), and (δ) at every vertex, all the arcs exit in the same direction (left or right)



142. [2.5] ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no arc (including its endpoints) lies strictly below another arc, and (δ) at every vertex, all the arcs exit in the same direction (left or right)



143. [1] sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

$$111{-}{-}{-} \qquad 11{-}1{-}{-} \qquad 11{-}{-}1{-} \qquad 1{-}11{-}{-} \qquad 1{-}1{-}1{-}$$

144. [1] sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$

111 112 113 122 123

145. [2] sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$

000 001 010 011 012

146. [1.5] sequences $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \leq 1$ and all partial sums are nonnegative

$$0,0 \qquad 0,1 \qquad 1,-1 \qquad 1,0 \qquad 1,1$$

147. [1.5] sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_i \geq -1$, all partial sums are nonnegative, and $a_1 + a_2 + \cdots + a_n = 0$

$$0,0,0 \quad 0,1,-1 \quad 1,0,-1 \quad 1,-1,0 \quad 2,-1,-1$$

148. [1.5] Sequences of $n-1$ 1's and any number of -1's such that every partial sum is nonnegative

$$1,1 \qquad 1,1,-1 \qquad 1,-1,1 \qquad 1,1,-1,-1 \qquad 1,-1,1,-1$$

149. [2.5] Sequences $a_1 a_2 \cdots a_n$ of nonnegative integers such that $a_j = \#\{i : i < j, a_i < a_j\}$ for $1 \leq j \leq n$

000 002 010 011 012

150. [2.5] Pairs (α, β) of compositions of n with the same number of parts, such that $\alpha \geq \beta$ (dominance order, i.e., $\alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i$ for all i)

$$(111,111) \quad (12,12) \quad (21,21) \quad (21,12) \quad (3,3)$$

151. [2] permutations $a_1 a_2 \cdots a_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that: (i) the first occurrences of $1, 2, \ldots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$

112233 112332 122331 123321 122133

152. [2.5] permutations $a_1 a_2 \cdots a_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called 321-*avoiding* permutations

123 213 132 312 231

153. [2] permutations $a_1 a_2 \cdots a_n$ of $[n]$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-*avoiding* permutations)

123 132 213 231 321

154. [2] permutations w of $[2n]$ with n cycles of length two, such that the product $(1,2,\ldots,2n)\cdot w$ has $n+1$ cycles

$$\begin{aligned}
(1,2,3,4,5,6)(1,2)(3,4)(5,6) &= (1)(2,4,6)(3)(5)\\
(1,2,3,4,5,6)(1,2)(3,6)(4,5) &= (1)(2,6)(3,5)(4)\\
(1,2,3,4,5,6)(1,4)(2,3)(5,6) &= (1,3)(2)(4,6)(5)\\
(1,2,3,4,5,6)(1,6)(2,3)(4,5) &= (1,3,5)(2)(4)(6)\\
(1,2,3,4,5,6)(1,6)(2,5)(3,4) &= (1,5)(2,4)(3)(6)
\end{aligned}$$

155. [2.5] pairs (u,v) of permutations of $[n]$ such that u and v have a total of $n+1$ cycles, and $uv=(1,2,\ldots,n)$

$$(1)(2)(3)\cdot(1,2,3) \quad (1,2,3)\cdot(1)(2)(3) \quad (1,2)(3)\cdot(1,3)(2)$$

$$(1,3)(2)\cdot(1)(2,3) \quad (1)(2,3)\cdot(1,2)(3)$$

156. [2] *noncrossing partitions* of $[n]$, i.e., partitions of $[n]$ such that if a,c appear in a block B and b,d appear in a block B', where $a<b<c<d$, then $B=B'$

123 12–3 13–2 23–1 1–2–3

(The unique partition of [4] that isn't noncrossing is 13–24.)

157. [2.5] noncrossing partitions of $[2n+1]$ into $n+1$ blocks, such that no block contains two consecutive integers

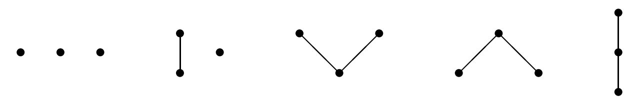
137–46–2–5 1357–2–4–6 157–24–3–6 17–246–3–5 17–26–35–4

158. [2.5] *nonnesting partitions* of $[n]$, i.e., partitions of $[n]$ such that if a,e appear in a block B and b,d appear in a *different* block B' where $a<b<d<e$, then there is a $c\in B$ satisfying $b<c<d$

123 12–3 13–2 23–1 1–2–3

(The unique partition of [4] that isn't nonnesting is 14–23.)

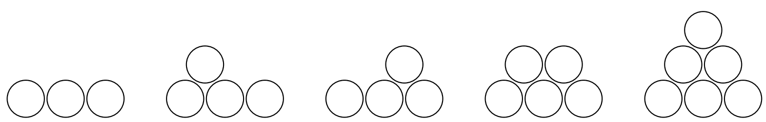
159. [2.5] nonisomorphic n-element posets (i.e., partially ordered sets) with no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$, where $\mathbf{a}+\mathbf{b}$ denotes the disjoint union of an a-element chain and a b-element chain



160. [2] relations R on $[n]$ that are reflexive (iRi), symmetric ($iRj \Rightarrow jRi$), and such that if $1 \leq i < j < k \leq n$ and iRk, then iRj and jRk (in the example below we write ij for the pair (i,j), and we omit the pairs ii)

$$\emptyset \quad \{12,21\} \quad \{23,32\} \quad \{12,21,23,32\} \quad \{12,21,13,31,23,32\}$$

161. [1.5] ways to stack coins in the plane, the bottom row consisting of n consecutive coins



162. [2.5] n-tuples $(a_1, a_2, \ldots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2 \cdots a_n1$, each a_i divides the sum of its two neighbors

$$14321 \quad 13521 \quad 13231 \quad 12531 \quad 12341$$

163. [3] n-element subsets S of $\mathbb{N} \times \mathbb{N}$ such that if $(i,j) \in S$ then $i \geq j$ and there is a lattice path from $(0,0)$ to (i,j) with steps $(0,1)$, $(1,0)$, and $(1,1)$ that lies entirely inside S

$$\{(0,0),(1,0),(2,0)\} \quad \{(0,0),(1,0),(1,1)\} \quad \{(0,0),(1,0),(2,1)\}$$

$$\{(0,0),(1,1),(2,1)\} \quad \{(0,0),(1,1),(2,2)\}$$

164. [3] positive integer sequences $a_1, a_2, \ldots, a_{n+2}$ for which there exists an

$$\begin{array}{ccccccccccccccccccccccccccc}
1 && 1 && 1 && 1 && 1 && 1 && 1 && 1 && 1 && 1 && 1 && 1 && 1 && 1 \\
& 1 && 3 && 2 && 1 && 5 && 1 && 2 && 3 && 1 && 3 && 2 && 1 && 5 & \\
&& 2 && 5 && 1 && 4 && 4 && 1 && 5 && 2 && 2 && 5 && 1 && 4 && \\
&&& 3 && 2 && 3 && 3 && 3 && 2 && 3 && 3 && 3 && 2 && 3 &&& \\
&&&& 1 && 5 && 2 && 2 && 5 && 1 && 4 && 4 && 1 && 5 &&&& \\
&&&&& 2 && 3 && 1 && 3 && 2 && 1 && 5 && 1 && 2 &&&&& \\
&&&&&& 1 && 1 && 1 && 1 && 1 && 1 && 1 && 1 &&&&&&
\end{array}$$

Figure 1: The frieze pattern corresponding to the sequence $(1, 3, 2, 1, 5, 1, 2, 3)$

integer array (necessarily with $n+1$ rows)

$$\begin{array}{ccccccccccccccccccc}
1 && 1 && 1 && \cdots && 1 && 1 && 1 && \cdots && 1 && 1 \\
& a_1 && a_2 && a_3 && \cdots && a_{n+2} && a_1 && a_2 && \cdots && a_{n-1} & \\
&& b_1 && b_2 && b_3 && \cdots && b_{n+2} && b_1 && \cdots && b_{n-2} && \\
&&&&&&&& \vdots &&&&&&&&&& \\
&&&& r_1 && r_2 && r_3 && \cdots && r_{n+2} && r_1 &&&& \\
&&&&& 1 && 1 && 1 && \cdots && 1 &&&&&
\end{array} \tag{6}$$

such that any four neighboring entries in the configuration $\begin{smallmatrix} & r & \\ s & & t \\ & u & \end{smallmatrix}$ satisfy $st = ru + 1$ (an example of such an array for $(a_1, \ldots, a_8) = (1, 3, 2, 1, 5, 1, 2, 3)$ (necessarily unique) is given by Figure 1):

12213 22131 21312 13122 31221

165. [3] n-tuples $(a_1, \ldots a_n)$ of positive integers such that the tridiagonal matrix

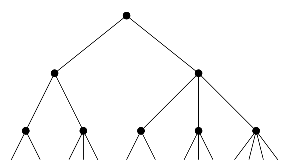
$$\begin{bmatrix}
a_1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
1 & a_2 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
0 & 1 & a_3 & 1 & \cdot & \cdot & \cdot & 0 & 0 \\
 & & & & & \cdot & & & \\
 & & & & & \cdot & & & \\
 & & & & & \cdot & & & \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & a_{n-1} & 1 \\
0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 & a_n
\end{bmatrix}$$

is positive definite with determinant one

NOTE. A real matrix A is *positive definite* if it is symmetric and every eigenvalue is positive; equivalently, A is symmetric and every leading principal minor is positive. A *leading principal minor* is the determinant of a square submatrix that fits into the upper left-hand corner of A.

131 122 221 213 312

166. [2] Vertices of height $n-1$ of the tree T defined by the property that the root has degree 2, and if the vertex x has degree k, then the children of x have degrees $2, 3, \ldots, k+1$



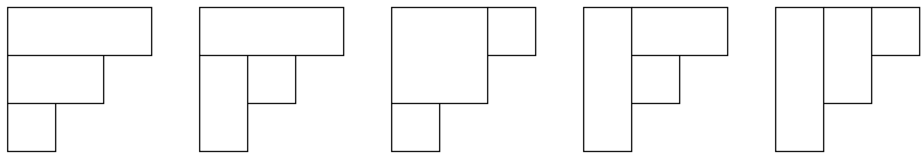
167. [2.5] Subsets S of $\mathbb{N}$ such that $0 \in S$ and such that if $i \in S$ then $i+n, i+n+1 \in S$

$$\mathbb{N}, \quad \mathbb{N}-\{1\}, \quad \mathbb{N}-\{2\}, \quad \mathbb{N}-\{1,2\}, \quad \mathbb{N}-\{1,2,5\}$$

168. [2] Ways to write $(1,1,\ldots,1,-n) \in \mathbb{Z}^{n+1}$ as a sum of vectors $e_i - e_{i+1}$ and $e_j - e_{n+1}$, without regard to order, where e_k is the kth unit coordinate vector in $\mathbb{Z}^{n+1}$:

$$\begin{gathered}(1,-1,0,0)+2(0,1,-1,0)+3(0,0,1,-1)\\(1,0,0,-1)+(0,1,-1,0)+2(0,0,1,-1)\\(1,-1,0,0)+(0,1,-1,0)+(0,1,0,-1)+2(0,0,1,-1)\\(1,-1,0,0)+2(0,1,0,-1)+(0,0,1,-1)\\(1,0,0,-1)+(0,1,0,-1)+(0,0,1,-1)\end{gathered}$$

169. [1.5] tilings of the staircase shape $(n, n-1, \ldots, 1)$ with n rectangles such that each rectangle contains a square at the end of some row



170. [2] $n \times n$ $\mathbb{N}$-matrices $M = (m_{ij})$ where $m_{ij} = 0$ unless $i = n$ or $i = j$ or $i = j - 1$, with row and column sum vector $(1, 2, \ldots, n)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

This concludes the list of objects counted by Catalan numbers. A few more problems related to Catalan numbers are the following.

171. (*) We have
$$\sum_{k=0}^{n} C_{2k} C_{2(n-k)} = 4^n C_n.$$

172. (*) An intriguing variation of Problem 170 above is the following. A bijective proof would be of great interest. Let $g(n)$ denote the number of $n \times n$ $\mathbb{N}$-matrices $M = (m_{ij})$ where $m_{ij} = 0$ if $i > j + 1$, with row and column sum vector $\left(1, 3, 6, \ldots, \binom{n+1}{2}\right)$. For instance, when $n = 2$ there are the two matrices
$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$
Then $g(n) = C_1 C_2 \cdots C_n$.

173. [2] (compare with Problem 168) Let $f(n)$ be the number of ways to write the vector
$$\left(1, 2, 3, \ldots, n, -\binom{n+1}{2}\right) \in \mathbb{Z}^{n+1}$$
as a sum of vectors $e_i - e_j$, $1 \le i < j \le n + 1$, without regard to order, where e_k is the kth unit coordinate vector in $\mathbb{Z}^{n+1}$. For instance, when $n = 2$ there are the two ways $(1, 2, -3) = (1, 0, -1) + 2(0, 1, -1) = (1, -1, 0) + 3(0, 1, -1)$. Assuming Problem 172, show that $f(n) = C_1 C_2 \cdots C_n$.

174. [2.5] The *Narayana numbers* $N(n,k)$ are defined by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

Let X_{nk} be the set of all sequences $w = w_1 w_2 \cdots w_{2n}$ of n 1's and n -1's with all partial sums nonnegative, such that

$$k = \#\{j : w_j = 1, w_{j+1} = -1\}.$$

Show that $N(n,k) = \#X_{nk}$. Hence by Problem 143, there follows

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

One therefore says that the Narayana numbers are a *refinement* of the Catalan numbers. There are many other interesting refinements of Catalan numbers, but we won't consider them here.