REGULI AND APPLICATIONS, ZARANKIEWICZ PROBLEM

One of our long-term goals is to pursue some questions of incidence geometry in $\mathbb{R}^3$. We recall one question to direct our focus during this lecture.

Question 1. *If $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$, and if at most 10 (or at most B) lines lie in a plane or degree 2 surface, what is the maximum possible number of intersection points of $\mathfrak{L}$?*

Now we introduce an important tool for incidence geometry in three dimensions.

1. REGULI

It is a classical fact that any three lines in $\mathbb{R}^3$ lie in the zero set of a degree 2 polynomial. This fact can be used to prove estimates in incidence geometry in three dimensions.

Proposition 1.1. *For any three lines l_1, l_2, l_3 in $\mathbb{R}^3$, there is a non-zero degree 2 polynomial Q that vanishes on all three lines.*

Proof. We will prove this result by counting dimensions. We can think of the argument as an example of the polynomial method.

Let $V(2)$ be the space of polynomials of degree ≤ 2 in three variables. The space $V(2)$ is a vector space of dimension 10. (A basis is given by $x^2, xy, xz, y^2, yz, z^2, x, y, z, 1$.)

Choose three points on each line. Let $p_{i,j}$ be three distinct points on l_i. By linear algebra, we can find a non-zero degree 2 polynomial Q that vanishes at all the points $p_{i,j}$. Since Q has degree 2 and vanishes at three distinct points of l_i, it must vanish on all of l_i. So Q vanishes on all three lines as desired. $\square$

This proposition allows us get good information about the lines that intersect all three lines l_1, l_2, and l_3. Exactly what happens depends a little on the properties of l_1, l_2, and l_3. Recall that two lines in $\mathbb{R}^3$ are skew if they don't intersect and they're not parallel. The most important case concerns three skew lines.

Proposition 1.2. *If l_1, l_2, and l_3 are pairwise skew, then there is an irreducible degree 2 algebraic surface $R(l_1, l_2, l_3)$ which contains every line that intersects l_1, l_2, and l_3.*

Proof. By the last proposition, there is a non-zero degree 2 polynomial Q that vanishes on l_1, l_2, and l_3. Let $R(l_1, l_2, l_3)$ be the zero set of Q. Suppose that l intersects l_1, l_2, and l_3. Since l_1, l_2, and l_3 are disjoint, the line l must intersect R in three distinct points. But then Q vanishes identically on l, and l is contained in R.

Finally, if Q was reducible, then it would be a product of linear factors, and R would be a union of two planes. But since the lines l_1, l_2, and l_3 are skew, no two of them lie in a plane, and so R cannot be a union of two planes. Also, if Q had degree 1, then R would be a plane, and this cannot happen either. $\square$

The surface $R(l_1, l_2, l_3)$ is called a regulus. Reguli have played an important role in incidence geometry for a long time... including the first work on the joints problem in the paper "Counting and cutting cycles of lines and rods in space", by Chazelle, Edelsbrunner, Guibas, Pollack, Seidel, Sharir, and Snoeyink (Computational Geometry, Theory and Applications 1 (1992) 305-323).

To complement this proposition, we record a couple of trivial lemmas which deal with the case when two lines are not skew.

Lemma 1.3. *Suppose that l_1 and l_2 are lines in $\mathbb{R}^3$ that intersect at a point p. Suppose that P is the plane that contains l_1 and l_2. Then any line which intersects both l_1 and l_2 either contains p or lies in P.*

Lemma 1.4. *Suppose that l_1 and l_2 are parallel. Let P be the plane that contains them. Then any line which intersects both l_1 and l_2 lies in P.*

Chazelle et al applied these results to 3d incidence geometry. For example, they proved that the number of joints determined by L lines is $\lesssim L^{7/4}$. We will use their method to work on Question 1.

Theorem 1.5. *Suppose that $\mathfrak{L}$ is a set of L lines in $\mathbb{R}^3$ with ≤ 10 lines in any plane or degree 2 surface. Then the number of intersection points of $\mathfrak{L}$ is $\lesssim L^{5/3}$.*

Proof. We will work with the intersection matrix of a set of lines $\mathfrak{L}$. Let us record which pairs of lines intersect. We make an $L \times L$ matrix A whose entries are 0 or 1, where the entry a_{ij} is 1 if and only if l_i and l_j intersect. (Convention for the diagonal: all zeroes.)

We write $|A|$ to mean the number of 1's in the matrix A. If every intersection point was a simple intersection of two lines, then the number of intersection points would be $(1/2)|A|$. Some intersection points have multiplicity 2, others may have very high multiplicity, and we want to keep track of them separately. We let A_t be the matrix with a 1 in the (i,j)-entry if l_i and l_j intersect at a point lying in $\sim 2^t$ lines of $\mathfrak{L}$. (More precisely, $\sim 2^t$ means more than 2^{t-1} but at most 2^t.) The number of points with intersection multiplicity $\sim 2^t$ is $\sim |A_t| 2^{-2t}$. Therefore, the number of intersection points is

$$\sim \sum_t |A_t| 2^{-2t}.$$

Our next goal is to estimate $|A_t|$.

Lemma 1.6. *Suppose that $\mathfrak{L}$ has ≤ 10 lines in any plane or degree 2 surface. Then A_t has no $3 \times 20 \cdot 2^t$ minor of all 1's.*

Proof. Suppose that A_t has a $3 \times 20 \cdot 2^t$ minor of all 1's. Let the three rows by labelled by l_1, l_2, l_3. If l_1, l_2, l_3 are all skew, then the ≥ 20 column lines all lie in the degree 2 surface $R(l_1, l_2, l_3)$, which is impossible. Suppose that l_1, l_2, l_3 are not all skew. After relabelling, we can assume that l_1 and l_2 are not skew. If l_1 and l_2 interesect in a point p and lie in a plane P, then we either get $5 \cdot 2^t$ column lines containing p or $5 \cdot 2^t$ column lines lying in P. There can't be that many lines in a plane. Also, by the definition of A_t, there should only be $\leq 2^t$ lines of $\mathfrak{L}$ containing p. Finally, if l_1 and l_2 are parallel lines in the plane P, then we get $10 \cdot 2^t$ column lines in the plane P. $\qquad\square$

Knowing that the matrix A_t does not have any $3 \times 20 \cdot 2^t$ minors of all 1's controls the number of 1's in the matrix by the following classical theorem. It touches on an important area of combinatorics with many open questions.

Theorem 1.7. *(Kővári-Sós-Turán, 1954) Suppose that A is an $L \times L$ matrix whose entries are 0 or 1. Suppose that A has no $V \times W$ minor of all 1's, for some integers $V \leq W$. Then the number of 1's in A is at most $C(V)W^{1/V}L^{\frac{2V-1}{V}}$.*

We give the proof and discuss the problem more generally below. So we see $|A_t| \lesssim 2^{t/3}L^{5/3}$.

$$\sum_t |A_t| 2^{-2t} \lesssim L^{5/3} \sum_t 2^{-(5/3)t} \lesssim L^{5/3}.$$

$\qquad\square$

Using the polynomial method, we will eventually improve this bound to $L^{3/2}$.

2. The Zarankiewicz problem

In the early 1950's, Zarankiewicz posed the following problem. Suppose that A is an $M \times N$ matrix with entries 0 or 1, and suppose that A has no $V \times W$ minor of all 1's. What is the maximum possible number of 1's that we can have in A? We considered the problem above for square matrices $M = N = L$.

Theorem 2.1. *(Kővári-Sós-Turán, 1954) Suppose that A is an $M \times N$ matrix whose entries are 0 or 1. Suppose that A has no $V \times W$ minor of all 1's, for some integers $V \leq W$. Then the number of 1's in A is at most $C(V)W^{1/V}MN^{\frac{V-1}{V}}$.*

Proof. Let $C_1, ..., C_N$ denote the columns of A. We can think of each column as a subset of the numbers $[1, ..., M]$. We let $\binom{M}{V}$ denote all of the sets of V distinct elements of the numbers $1, ..., M$. We let $\binom{C_j}{V}$ denote all of the sets of V distinct

elements of C_j. Clearly $\binom{C_j}{V} \subset \binom{M}{V}$. We let $|C_j|$ be the number of elements in C_j, so that the number of elements in $\binom{C_j}{V}$ is $\binom{|C_j|}{V}$.

The condition that A has no $V \times W$ minor of all 1's implies that each element of $\binom{M}{V}$ occurs in $< W$ of the sets $\binom{C_j}{V}$. So we get the following inequality:

$$\sum_{j=1}^{N} \binom{|C_j|}{V} < W \binom{M}{V}.$$

We write $A \lesssim B$ for $A \leq C(V)B$. Up to constant $C(V)$, the left-hand side is roughly $\sum |C_j|^V$, and so

$$\sum_{j=1}^{N} |C_j|^V \lesssim W M^v.$$

The total number of 1's in A is $\sum_j |C_j|$. Now by Holder's inequality,

$$\sum_{j=1}^{N} |C_j| \leq (\sum |C_j|^V)^{1/V} N^{\frac{V-1}{V}} \lesssim (WM^V)^{1/V} N^{\frac{V-1}{V}} = W^{1/V} M N^{\frac{V-1}{V}}.$$

$\square$

The Zarankiewicz problem has been in the background of many questions in incidence geometry. For example, suppose that we have S points and L lines in the plane. We can form the incidence matrix, an $S \times L$ matrix. Each row corresponds to a point, each column corresponds to a line, and there is a 1 if the point lies on the line. This matrix has no 2×2 submatrix of all 1's, because two lines intersect in only one point. Therefore, the number of incidences is $\lesssim SL^{1/2}$. (The situation is symmetric, so the number of incidences is also $\lesssim S^{1/2}L$.) These bounds are equivalent to the first bounds we proved about incidence geometry of points and lines in the plane.

We can do other examples with unit circles and/or circles. For instance, consider the unit distance problem. We have N points in the plane. Form an $N \times N$ 0/1 matrix with a 1 whenever the distance between the corresponding points is 1. We can think of this as an incidence matrix. The rows correspond to points, and the columns correspond to unit circles centered at the points, and an entry is 1 if the point corresonding to the row lies on the unit circle corresponding to the column. Two unit circles intersect in at most 2 points, and so this matrix has no 3×2 minor of all 1's. Therefore, the number of unit distances is $\lesssim N^{3/2}$.

It's a very interesting question how sharp the Kővári-Sós-Turán theorem is. After a couple of examples, we come to deep open problems.

Example 1. Consider an $N \times N$ 0-1 matrix with no 2×2 submatrix. The KST theorem says that the matrix has $\lesssim N^{3/2}$ 1's. This estimate is sharp. The example was discovered by Reiman in 1958 ("Uber ein Problem von K. Zarankiewicz", Acta Mathematica Hungarica 9 (34): 269-273.) We have essentially already seen the example: it is the incidence matrix of lines over a finite field. We pick $N = q^2$ lines in the plane $\mathbb{F}_q^2$. We let the rows of our matrix correspond to the points of $\mathbb{F}_q^2$ and the columns correspond to the q^2 chosen lines. We put a 1 in the matrix if the point corresponding to the row lies in the line corresponding to the column. Since two lines intersect in at most 1 point, there are no 2×2 submatrices. Since each line contains q points, our matrix has $q^3 = N^{3/2}$ 1's. Working with the projective plane over $\mathbb{F}_q$ is even slightly better.

Example 2. Next consider an $N \times N$ 0-1 matrix with no 3×3 submatrix. The KST theorem says that the matrix has $\lesssim N^{5/3}$ 1's. This example was found by Brown in the early 60's. It involves a clever construction with some low-degree polynomials over finite fields. We will do it on the homework.

This example is clever and special, and it seems very hard to generalize it to 4×4 minors. Consider an $N \times N$ 0-1 matrix with no 4×4 submatrix. The KST theorem says that the matrix has $\lesssim N^{7/4}$ 1's. But the best examples have only $\sim N^{5/3}$ 1's - and these are basically Brown's examples which have no 3×3 minor of all 1's! It's a longstanding open problem in combinatorics where the truth lies between Brown's example and the KST upper bound.

Example 3. Finally, consider an $N \times N$ 0-1 matrix with no $V \times V$ submatrix. The KST theorem says that the number of 1's is $\leq C(V)N^{2-(1/V)}$. For large V, the best examples come from a random construction. On the homework, you will use the technique to prove that there are examples with $\sim N^{2-\frac{2}{V+1}}$ 1's. For $V \geq 5$, I believe these are the largest known examples.

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