SPECIAL POINTS AND LINES OF ALGEBRAIC SURFACES

1. Introduction

As we have seen many times in this class we can encode combinatorial information about points and lines in terms of algebraic surfaces. Looking at these surfaces can in turn show us things about the underlying combinatorics that are not obvious at first glance.

One theme that comes up often when looking at these surfaces is that they contain special points– such as critical points, etc- whose presence or absence tells us something about our underlying combinatorial problem. Todays lecture focuses on this theme.

2. Critical points

Consider a polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$. Let $Z(P)$ be its zero set, aka

$$Z(P) = \{x \in \mathbb{R}^n | P(x) = 0\}$$

Recall that a point x is called a critical point if and only if $\nabla P(x) = 0$. Recalling the implicit function theorem:

Theorem 2.1. *If a point $x \in Z(P)$ is not a critical point of P then $Z(P)$ is a smooth manifold in some open neighborhood centered at x.*

We might hope that, in general, there are very few critical points. This is not always the case. Consider

$$P(x_1, \ldots, x_n) = x_1^2$$

Then every point on the plane $\{x_1 = 0\}$ is a critical point of P. Unfortunate as this may be it turns out that, in some sense, this is the only case we have to worry about.

Definition 2.2. *A polynomial P is square free if, whenever a polynomial Q^2 divides P, we have that Q is constant.*

Our pathological example is not square free– if, however, we only consider square free polynomial then everything works out:

Lemma 2.3. *If P is square free then $P, \partial_1 P, \ldots, \partial_n P$ have no common nonconstant factors.*

Proof. Assume that P is square free, and that $P, \partial_1 P, \ldots, \partial_n P$ have a common non-constant factor Q. Without loss of generality we can assume that this factor is irreducible. We say that $R \sim Q$ if $R = cQ$ for some $c \neq 0$. Then note that we can write

$$P = \Pi P_j$$

where each of the P_j is irreducible. Since Q is irreducible and divides P it must be the case that $Q \sim P_j$ for some j. This implies P_j divides $\partial_i P$ for all i. Using the product rule note that

$$\partial_i P = \sum_{j_0} \partial_i P_{j_0} \prod_{j \neq j_0} P_j$$

Therefore, since P_j divide $\partial_i P$ for all i, the fact that P_j divides $\prod_{k \neq j_0} P_k$ for all $j_0 \neq j$ tells us that P_j divides $\partial_i P_j \prod_{j \neq k} P_k$. Since P is square free we know that P_j can not divide $\prod_{j \neq k} P_k$. This implie P_j must divide $\partial_i P_j$. However, since $\partial_i P_j$ has smaller degree than P_j it must be that $\partial_i P_j = 0$ for all i. This implies P_j, and thus Q, is constant. Therefore we have a contradiction. □

This can be used to prove the following result.

Proposition 2.4. *If $n = 2$ and P is a square free polynomial of degree d then the number of critical points in $Z(P)$ is at most $2d^2$.*

Proof. Assume $x \in Z(P)$ is a critical point, then $P(x) = 0$ and $\partial_i P(x) = 0$ for all i. We can write $P = \prod P_j$, where P_j is irreducible. The product formula then implies that either $x \in Z(P_j)$ for at least two j, or else x is a critical point of $Z(P_j)$ for some j. We call these points type a and type b respectively.

First let us count the number of type a points. Since each pair P_i, P_j is relatively prime (assuming $i \neq j$) the number of x that are zeros of of both P_i and P_j is, according to Bezouts theorem, bounded above by $deg(P_i)deg(P_j)$. Therefore the total number of type a points is bounded above by

$$\sum_{i \neq j} deg(P_i)deg(P_j) \leq d^2$$

Next we want to bound the number of type b points. We know that P_j is irreducible, and that there exists i so that $\partial_i P_j$ is not identically 0. Since P_j is irreducible we know that P_j and $\partial_i P_j$ can have no common factor, so by Bezouts they share at most

$$deg(P_j)deg(\partial_i P_j) \leq deg(P_j)^2$$

zero points, so the number of critical points in $Z(P_j)$ is at most $deg(P_j)^2$, so the total number of type b points is at most

$$\sum_j deg(P_j)^2 \leq d^2$$

Adding these bounds together gives us our result. □

The above theorem relied heavily on Bezouts theorem. In some cases we need a slightly different version of Bezouts, one that works for lines instead of points.

Theorem 2.5. *If $P, Q \in \mathbb{R}[x_1, x_2, x_3]$ have no common factor then the set $Z(P) \cap Z(Q)$ contains at most $deg(P)deg(Q)$ lines*

A proof of the above is given in the notes from last time. To gain some intuition of why it is true assume we pick some random plane Π in $\mathbb{R}^3$. We can then restrict P and Q to Π, giving us $\tilde{P}$ and $\tilde{Q}$. In general we expect each line in $Z(P, Q) = Z(P) \cap Z(Q)$ to intersect Π exactly once, so if $Z(P, Q)$ contains L lines then

$$L \leq |Z(P, Q) \cap \Pi| = |Z(\tilde{P}, \tilde{Q})|$$

we expect that $\tilde{P}$ and $\tilde{Q}$ won't have any factors in common (though we will not prove this here), so Bezout tells us that

$$L \leq |Z(\tilde{P}, \tilde{Q})| \leq deg(\tilde{P})deg(\tilde{Q}) \leq deg(P)deg(Q)$$

which is what we wanted.

Now that we have a version of Bezout for lines we can use an almost identical argument to the one in Proposition 2.4 to give us

Proposition 2.6. *If $P \in \mathbb{R}[x_1, x_2, x_3]$ has degree d and is square free then $Z(P)$ contains at most $2d^2$ critical lines (aka lines all of whose points are critical).*

3. Joints and critical points

We saw a connection between combinatorics and critical points in the joints problem. Below is a sketch of an alternative proof of the joints problem (the original one) that makes this even more explicit. The bound for the joints problem follows easily from the below lemma by induction:

Lemma 3.1. *If we have L lines in $\mathbb{R}^3$ one of these lines has at most $1000L^{\frac{1}{2}}$ joints on it.*

Proof. This is only a sketch.

We will proceed in a proof by contradiction.

The first step is degree reduction: create a polynomial P, $P = 0$ on all the lines. We can easily choose P to be square free, with $deg(P) \leq \frac{1}{10}L^{\frac{1}{2}}$

Step 2 is to note that $\nabla P = 0$ on all the joints. Step 3 is to note, since we are assuming each of the lines has a lot of joints on it that $\nabla P = 0$ identically on all lines.

This implies that $Z(P)$ has L critical lines in it, were $L > 2deg(P)^2$. This, however, is a contradiction. □

4. Flat Points

In the below assume that $n = 3$.

Definition 4.1. *Assume that $x \in Z(P)$ is not a critical point. We can rotate and translate $Z(P)$ so that $x = 0$ and*

$$T_x Z(P) = \{x_3 = 0\}$$

(where $T_x M$ is the tangent plane of M at x for a given x and M). Then around x the surface $Z(P)$ is given by the equation

$$x_3 = Q(x_1, x_2) + O(x_1, x_2)^3$$

where Q is homogeneous of degree 2. We say that x is flat iff $Q = 0$.

There are numerous equivalent ways to state this condition. Let N be the normal to $Z(P)$. Then x is flat iff $\partial_v N(x) = 0$ for all $v \in T_x Z(P)$.

Note that $N = \frac{\nabla P}{|\nabla P|}$. More than that $v \in T_x Z(P)$ iff $v \cdot \nabla P = 0$. Therefore we can use this to try to give an alternative characterization of flat. Before doing so, however, we want to introduce some tricks.

The first trick is to see that $\partial_v N = 0$ iff $\partial_v \nabla P$ is parallel to ∇P, which happens iff

$$\partial_v \nabla P \times \nabla P = 0$$

The second trick is to note that $\{e_j \times \nabla P\}_{j=1,2,3}$ is a spanning subset of $T_x Z(P)$.

The above inspires us to define

$$SP(x) = \{(\partial_{e_j \times \nabla P} \nabla P(x)) \times \nabla P(x)\}_{j=1,2,3}$$

Note that $SP(x)$ is a collection of 3 vectors, and is polynomial in x. We can then put the above together in a proposition:

Proposition 4.2. *If $x \in Z(P)$ then $SP(x) = 0$ iff $\nabla P(x) = 0$ or x is flat.*

Proof. This just follows by stringing all the above results together.

Clearly if $\nabla P(x) = 0$ then $(\partial_{e_j \times \nabla P} \nabla P(x)) \times \nabla P(x) = 0$ so $SP(x) = 0$. Similarly if x is flat then

$$(\partial_{e_j \times \nabla P} \nabla P(x)) \times \nabla P(x) = 0$$

so $SP(x) = 0$.

On the other hand assume that $SP(x) = 0$ and that $\nabla P(x) \neq 0$ then since

$$(\partial_{e_j \times \nabla P} \nabla P(x)) \times \nabla P(x) = 0$$

for all $j = 1, 2, 3$, and since $\{e_j \times \nabla P\}_{j=1,2,3}$ spans $T_x Z(P)$ it must be that

$$(\partial_v \nabla P(x)) \times \nabla P(x) = 0$$

for all $v \in T_x Z(P)$, which implies

$$(\partial_v \nabla P(x)) \times N = 0$$

which implies x is flat. □

We can see that, if every point of $Z(P)$ is flat then $Z(P)$ is a plane. This, however, is the only case when we can have a lot of flat lines.To see this, define a special line be one where every point on the line satisfies $SP(x) = 0$ (note that ever flat line is special).

Proposition 4.3. *If P is an irreducible degree d polynomial and P is not linear then $Z(P)$ contains at most $3d^2$ special lines.*

Proof. First consider the case that P does not divide every component of $SP(x)$, which is to say that there is some j so that P does not divide some component of $(\partial_{e_j \times \nabla P} \nabla P) \times \nabla P$. The claim then follows by Bezout.

Therefore assume P divides every component of $SP(x)$. This implies $SP(x) = 0$ on all of $Z(P)$.

First consider the case that there exists a noncritical point $x \in Z(P)$. This implies $Z(P)$ is smooth in a neighborhood of x. More than that it is flat in that neighborhood, so it must locally be a plane. Therefore $Z(P)$ must contain that entire plane. This implies that the linear polynomial whose zero set equals this plane must divide P. Putting this together with the fact that P is irreducible implies that $Z(P)$ must be linear. This is a contradiction.

Therefore all points in $Z(P)$ are critical. Therefore every line is critical, but we know there are at most $2d^2$ critical lines, so we are done. □

We will apply the above to prove some combinatorial results next week. In particular we will prove a special case of the ES conjecture, namely that:

Theorem 4.4. *Assume we have L lines in $\mathbb{R}^3$, with at most B lines on any plane. Then if P_3 is the set of points lying on 3 lines we have that $|P_3| \leq cBL$ for some constant c*

If we define P_k in the analogous way then $|P_k| \leq cL^{\frac{3}{2}} k^{-2}$.