2.001 - MECHANICS AND MATERIALS I
Lecture #16
11/6/2006
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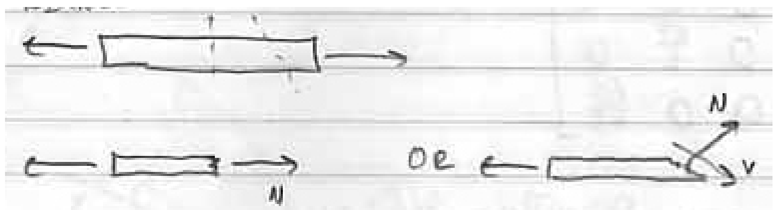
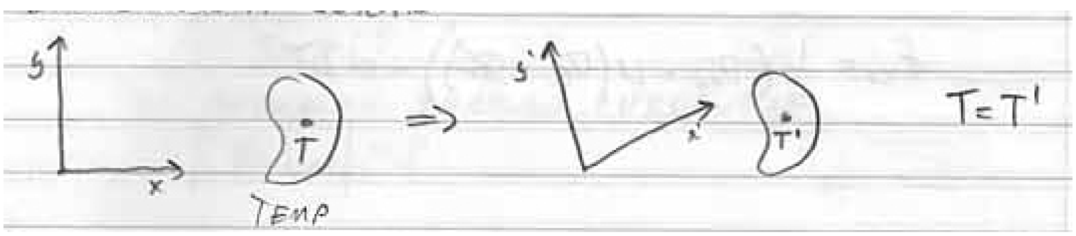
Transformations

Recall:

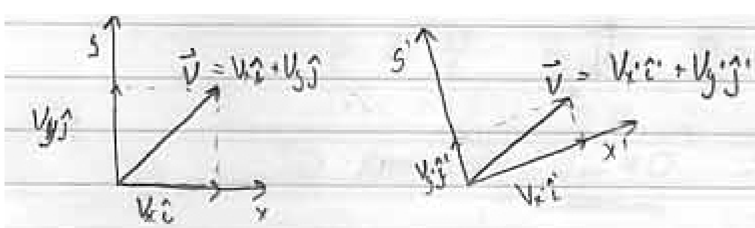


: Stress Tensor has new rules for transformations.

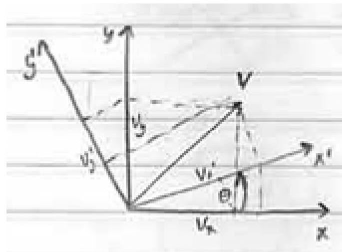
Transforming a scalar:



Transforming a vector:



Transformation Equations:

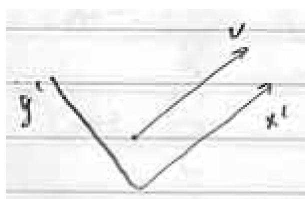


$$v'_x = v_x \cos\theta + v_y \sin\theta$$
$$v'_y = v_x \sin\theta + v_y \cos\theta$$

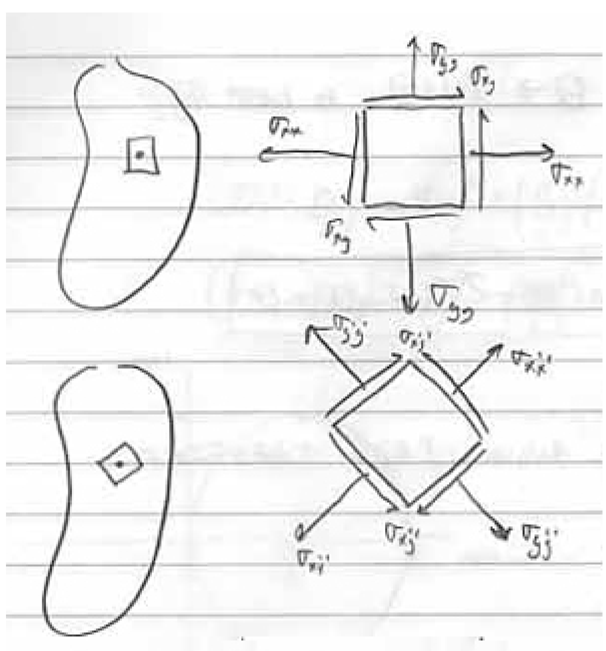
Note:

$$v_x^2 + v_y^2 = v^2 = v_x^{'2} + v_y^{'2}$$

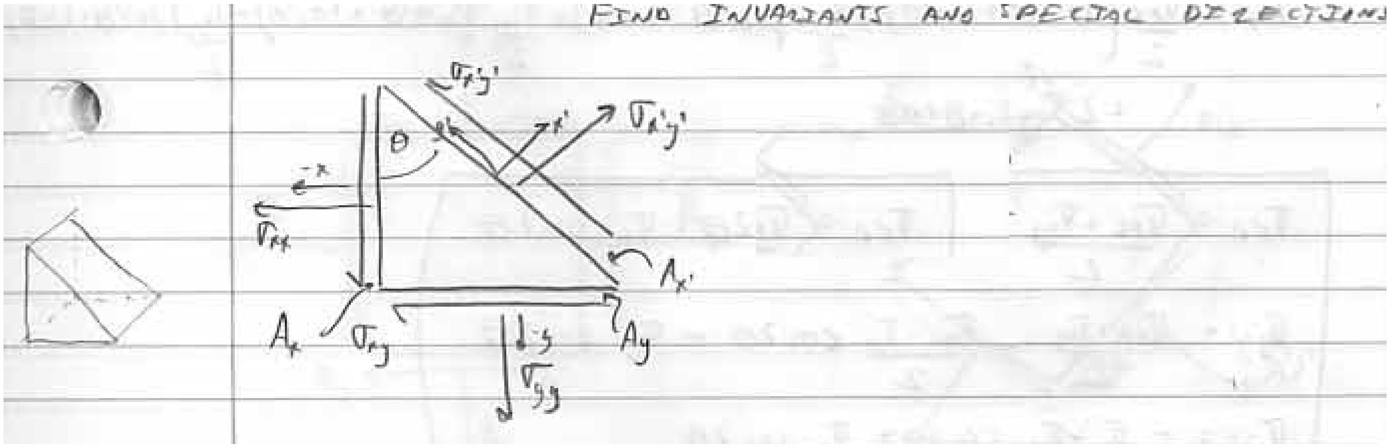
Invariant (Quantity is independent of frame)



Special Direction



Find how σ_{xx},σ_{yy},σ_{xy} $\Leftrightarrow$ $\sigma_{x'x'}$,$\sigma_{y'y'}$,$\sigma_{x'y'}$



$$A_x = A_{x'} \cos\theta$$
$$A_y = A_{x'} \sin\theta$$

$$\sum F_{x'} = 0$$

$$\sigma_{x'x'} A_{x'} - \sigma_{yy} \sin\theta A_y - \sigma_{xy} \cos\theta A_y - \sigma_{xx} \cos\theta A_x - \sigma_{xy} \sin\theta A_x = 0$$

Substitute A_x, A_y:

$$\sigma_{x'x'} A_{x'} - \sigma_{yy} \sin\theta \sin\theta A_{x'} - \sigma_{xy} \cos\theta \sin\theta A_{x'} - \sigma_{xx} \cos\theta \cos\theta A_{x'} - \sigma_{xy} \sin\theta \cos\theta A_{x'} = 0$$

So:

$$\sigma_{x'x'} = \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\sigma_{xy} \cos\theta \sin\theta$$

$$\sum F_{y'} = 0$$
$$\sigma_{x'y'} A_{x'} - \sigma_{yy} \cos\theta A_y + \sigma_{xy} \sin\theta A_y + \sigma_{xx} A_x \sin\theta - \sigma_{xy} A_x \cos\theta = 0$$
$$\sigma_{x'y'} = (\sigma_{yy} - \sigma_{xx}) \cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)$$

Use $\sigma_{x'x'}$ equation with $\theta \to 90 + \theta$ to get $\sigma_{y'y'}$

$$\sin(\theta + 90^\circ) = \cos\theta$$
$$\cos(\theta + 90^\circ) = -\sin\theta$$

$$\sigma_{y'y'} = \sigma_{xx} \sin^2\theta + \sigma_{yy} \cos^2\theta - 2\sigma_{xy} \sin\theta \cos\theta$$

Rewrite using "double angle" trig. identities:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\cos^2\theta + \sin^2\theta = 1$$

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos(2\theta) + \sigma_{xy}\sin(2\theta)$$

$$\sigma_{y'y'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos(2\theta) - \sigma_{xy}\sin(2\theta)$$

$$\sigma_{x'y'} = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin(2\theta) + \sigma_{xy}\cos(2\theta)$$

EXAMPLE: Uniaxial Loading

$$\sigma_{xy} = \sigma_{yy} = 0$$

So:

$$\sigma_{x'x'} = \frac{\sigma_{xx}}{2} + \frac{\sigma_{xx}}{2}\cos(2\theta) = \frac{\sigma_0}{2} + \frac{\sigma_0}{2}\cos(2\theta)$$

$$\sigma_{y'y'} = \frac{\sigma_{xx}}{2} - \frac{\sigma_{xx}}{2}\cos(2\theta) = \frac{\sigma_0}{2} - \frac{\sigma_0}{2}\cos(2\theta)$$

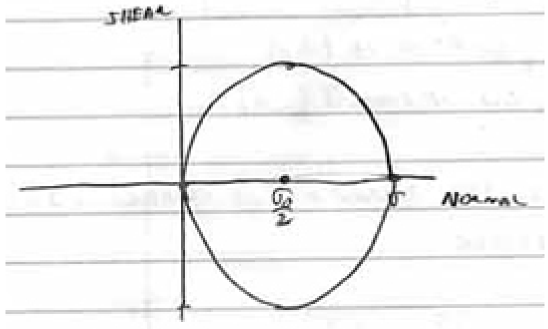
$$\sigma_{x'y'} = -\frac{\sigma_{xx}}{2}\sin(2\theta) = \frac{sigma_0}{2}\sin(2\theta)$$

Square and add them:

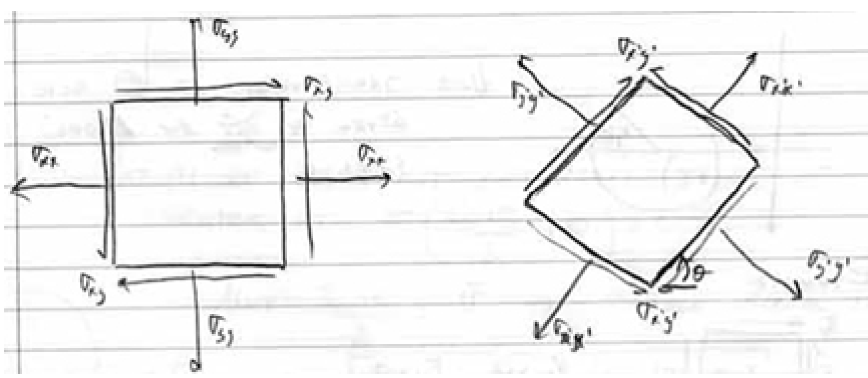
$$(\sigma_{x'x'} - \frac{\sigma_0}{2})^2 + \sigma_{x'y'}^2 = \left(\frac{\sigma_0}{2}^2\right)(\cos^2 2\theta + \sin^2 2\theta)$$

$$(\sigma_{x'x'} - \frac{\sigma_0}{2})^2 + \sigma_{x'y'}^2 = \left(\frac{\sigma_0}{2}^2\right)$$

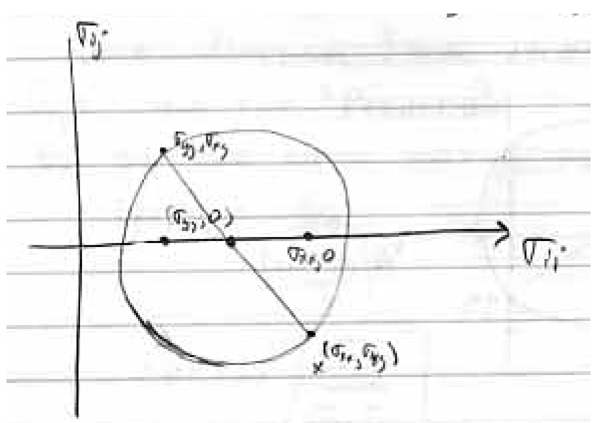
: Note: This is a circle. $(x-a)^2 + y^2 = r^2$



This is an example of Mohr's circle.



Draw for $\sigma_{xx} > \sigma_{yy} > 0$;$\sigma_{xy} > 0$.



Procedure to draw Mohr's circle:

For x-face

1. For positive σ_{xx}, go to right of (0,0).
2. For positive σ_{xy}, go down from (σ_{xx},0).
3. Plot point.

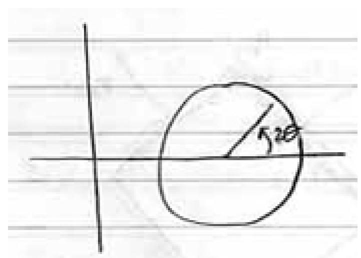
For y-face

1. For positive σ_{yy}, go right of (0,0).
2. For positive σ_{xy}, go up from (σ_{yy},0).
3. Plot point.

Connect points, this is the diameter of Mohr's circle.

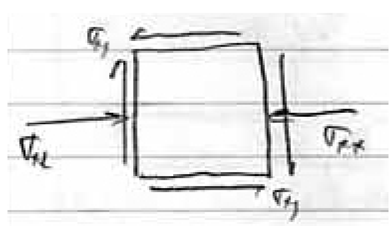
Finish drawing circle.

To use Mohr's Circle:



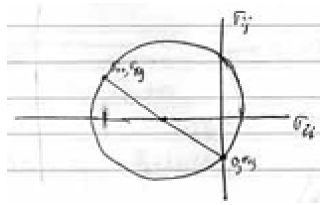
Note: Transformations of θ are given by 2θ on Mohr's circle.

EXAMPLE:



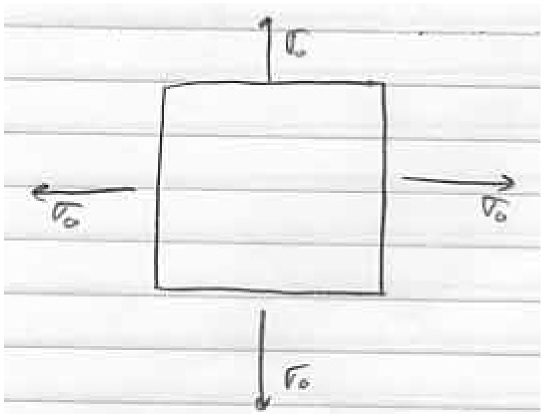
$$\sigma_{xx} < 0$$

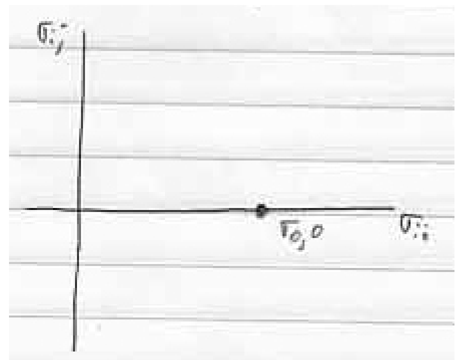
$$\sigma_{xy} < 0$$



$$\sigma_{yy} = 0$$

EXAMPLE: Hydrostatic Pressure

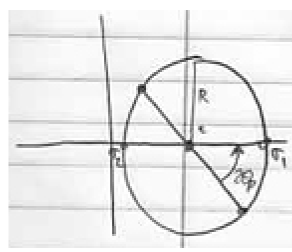




$$\sigma_{xx} = \sigma_{yy} = \sigma_0$$

$$\sigma_{xy} = 0$$

Points of interest on Mohr's circle



Center:

$$\left(\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$$

Note: $\frac{\sigma_{xx}+\sigma_{yy}}{2} = \overline{\sigma} \Rightarrow$ Average normal Stress

Radius:

$$\sqrt{\sigma_{xy}^2 + (\sigma_{xx}-\overline{\sigma})^2} = \sqrt{\sigma_{xy}^2 + \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2} = R$$

Note: Maximum shear stress = R

σ_1,σ_2 are the principal stresses.

θ_p is the angle of the principal direction.

$$\tan\theta_p = \frac{\sigma_{xy}}{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)}$$

$$\theta_p = \tan^{-1}\left[\frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}}\right]$$