Problem 1 (20 points)
Part A:

Because the beam is still in elastic region, the stress field can be expressed as:

$$\sigma(y) = -\frac{yM_y}{I} \tag{1}$$

Since the most highly stressed region is at the verge of yielding, we have

$$|\sigma(y)_{max}| = |-\frac{yM_y}{I}|_{y=\pm h/\sqrt{2}} = \sigma_y \Rightarrow M_y = \frac{\sigma_y I}{h/\sqrt{2}} \tag{2}$$

For the diamond-orientation beam, I is calculated as:

$$I = \int y^2 dA = 2\int_0^{\frac{h}{\sqrt{2}}} (\frac{h}{\sqrt{2}} - y) \times 2dy = \frac{h^4}{12} \tag{3}$$

The area moment of inertia for this diamond-orientation is the same as the cross-section appears as a square. Substitution of I into Eq. 2, we get that:

$$M_{y(diamond)} = \frac{\sqrt{2}\sigma_y h^3}{12} \tag{4}$$

Part B:

$$M_{L(diamond)} = \int \sigma_y(-y)dA = 2\int_0^{\frac{h}{\sqrt{2}}} \sigma_y y \times (\frac{h}{\sqrt{2}} - y) \times 2dy = \frac{\sqrt{2}\sigma_y h^3}{6} \tag{5}$$

Part C:

$$\frac{M_{L(diamond)}}{M_{y(diamond)}} = \frac{\frac{\sqrt{2}\sigma_y h^3}{6}}{\frac{\sqrt{2}\sigma_y h^3}{12}} = 2 \tag{6}$$

When the cross-section appears as a square, M_y is calculated as:

$$M_{y(square)} = \frac{\sigma_y I}{y_{max}} = \frac{\sigma_y h^4/12}{h/2} = \frac{\sigma_y h^3}{6} \tag{7}$$

and M_L is:

$$M_{L(square)} = \int \sigma(-y)dA = 2 \int_0^{\frac{h}{2}} \sigma_y y \times h dy = \frac{\sigma_y h^3}{4} \tag{8}$$

So the ratio of M_L/M_y for square-orientation is:

$$\frac{M_{L(square)}}{M_{y(square)}} = \frac{\frac{\sigma_y h^3}{4}}{\frac{\sigma_y h^3}{6}} = \frac{3}{2} \tag{9}$$

Part D:

The ratio of the first-yield bending moments for the two orientations is:

$$\frac{M_{y(diamond)}}{M_{y(square)}} = \frac{\frac{\sqrt{2}\sigma_y h^3}{12}}{\frac{\sigma_y h^3}{6}} = 1/\sqrt{2} \tag{10}$$

The first-yield bending moment is calculated by:

$$M_y = \frac{\sigma_y I}{y_{max}} \tag{11}$$

Since σ_y is a constant, and I is the same for these two orientations, the above ratio is only determined by the ratio of y_{max}. The ratio of limit moment for the two orientations is:

$$\frac{M_{L(diamond)}}{M_{L(square)}} = \frac{\frac{\sqrt{2}\sigma_y h^3}{6}}{\frac{\sigma_y h^3}{4}} = \frac{2}{3}\sqrt{2} < 1.0 \tag{12}$$

The limit moment for a symmetrical cross-section is calculated by:

$$M_L = 2\sigma_y \int_{A/2} y dA \tag{13}$$

where $A/2$ is the 1/2 area of the cross section in which $y \geq 0$. For the diamond-orientation, though the maximum value of y is $\sqrt{2}$ times larger than the square-orientation, the major part of its cross-section is located at the region with small y coordinate. Thus the ratio of limit moment calculated above is smaller than one.

Since $M_{y(diamond)} < M_{y(square)}$ and $M_{L(diamond)} < M_{L(square)}$, we should use the beam in the square-orientation.

Part E:

The unloaded stress field is expressed as:

$$\sigma_{unloaded} = \sigma_{loaded} + \Delta\sigma(y) \tag{14}$$

With the assumption of elastic unloading from the limit condition M_L, $\Delta\sigma(y)$ can be expressed as following:

$$\Delta\sigma(y) = \frac{-y\Delta M}{I} = \frac{yM_{L(diamond)}}{I} \tag{15}$$

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Substitution of $\Delta\sigma(y)$, I and $M_{L(diamond)}$ into Eq. 14, we find that:

$$\sigma_{unloaded} = \left\{ \begin{array}{ll} -\sigma_y + 2\sqrt{2}\sigma_y(y/h) & \text{if } y \geq 0; \\ \sigma_y + 2\sqrt{2}\sigma_y(y/h) & \text{if } y < 0. \end{array} \right. \tag{16}$$

Similarly, the unloaded stress field for the square-orientation is expressed as:

$$\sigma_{unloaded} = \sigma_{loaded} + \frac{yM_{L(square)}}{I} = \left\{ \begin{array}{ll} -\sigma_y + 3\sigma_y(y/h) & \text{if } y \geq 0; \\ \sigma_y + 3\sigma_y(y/h) & \text{if } y < 0. \end{array} \right. \tag{17}$$

The stress field is shown in Figure 1.

For the diamond-orientation, since the residual stress at $y = y_{max}$ is σ_y, no negative moment (which will cause positive stress increment to the upper part and negative increment of stress for the lower part) can be applied without further plasticity.
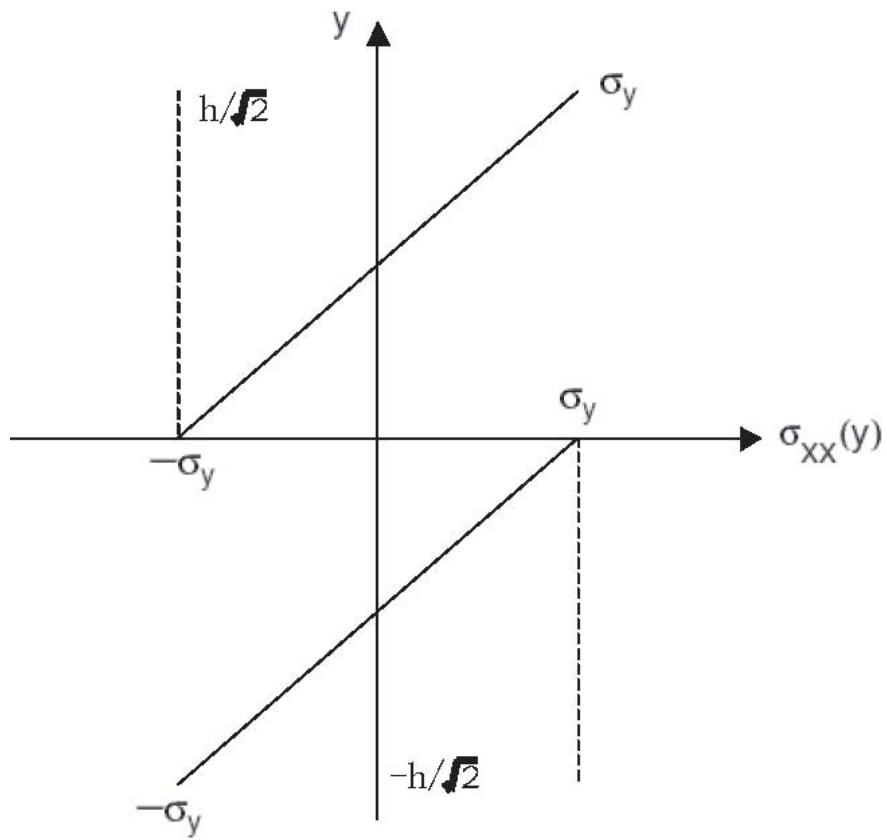
(a) diamond-orientation
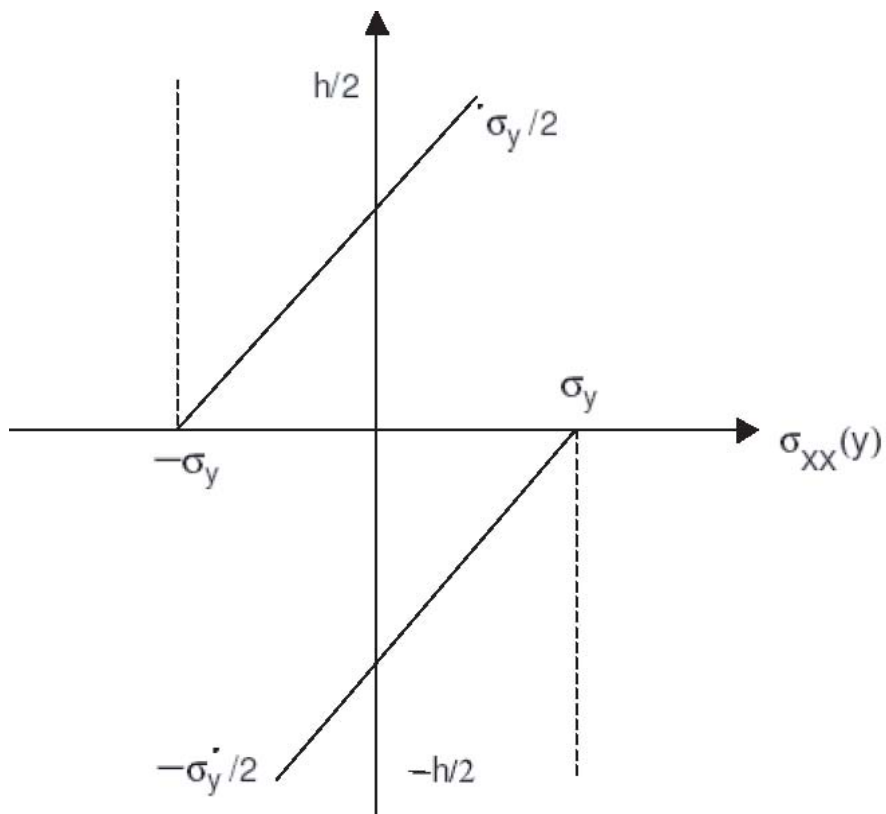
(b) square-orientation

Figure 1 Unloaded stress

Problem 2 (30 points)

There are 4 basic ways to strengthen metallic crystalline materials. All 5 mechanisms impede dislocation motion by creating an impedance to dislocation motion which raises the shear stress required for dislocation motion. Three of the mechanisms involve objects that get in the way of dislocation motion. In those mechanisms, the shear strength has the general relation $\tau = \frac{Gb}{L}$ where L is some characteristic length between objects.

1. One mechanism is obstacle/precipitate strengthening. The shear stress required for a dislocation to loop around an obstacle or second phase precipitate in the matrix is a function of the precipitate spacing or diameter L_0 is precipitate spacing, D_0 is precipitate diameter, c_0 is alloy concentration:

$$\tau = \frac{Gb}{L_0}$$

$$(\frac{D_0}{L_0})^3 \sim c_0$$

$$\tau = \frac{Gbc_0^{1/3}}{D_0}$$

So, smaller diameters increase the shear stress required for dislocation motion, thus "smaller diameters are stronger." This is valid down to the diameter where the dislocation begins to cut through the particle, at which point the shear stress begins to decrease with diameter.

2. A second strengthening mechanism is solid solution strengthening (SSS). In SSS, alloying atoms fill either substitutional positions or interstitial positions in the lattice. In a solid solution of concentration c_0, the spacing of dissolved atoms on the slip plane L varies as $c_0^{-1/2}$. Since, $\tau = \frac{Gb}{L}$, we have $\tau \propto c_0^{1/2}$. So, the greater the concentration the 'rougher' the slip plane, which reveals "Smaller defect spacing is stronger."

3. A third strengthening mechanism is strain hardening. When dislocations meet each other during plastic deformation they impede each other and also multiply. More plastic strain means more dislocations are created. Because a large shear stress is required to have one dislocation pass through another, a higher dislocation density means the shear stress required for dislocation motion should be higher. Thus if dislocation density is higher, the space between dislocations is smaller. Therefore having "smaller spaces between dislocations is stronger." Mathematically, this is seen by the dislocation density, ρ.

$$\rho = \frac{L_d}{V}$$

where L_d is the dislocation line length in a given volume and V is the volume. The spacing between dislocations can be thought of as $L_{ds} = 1/\sqrt{\rho}$. Thus the distance between other dislocations is L_{ds} and $\tau = \frac{Gb}{L_{ds}} = Gb\sqrt{\rho}$

4. A final strengthening mechanism fitting into the "smaller is stronger" category is grain size strengthening. Single crystalline grains are randomly aligned and misoriented with each other. Thus, at a boundary, the dislocation has a problem entering the next grain. The smaller the grain size, the more boundaries exist, therefore creating high impedance to dislocation motion. Again, "smaller grain size is stronger." The relation between grain size and strength is known as the Hall-Petch relation, $\sigma_y = \sigma_0 + Kd^{1/2}$

Problem 3 (30 points)

PART A: The elastic strain is not important because we are talking about a very large deformation ($H/H_0 = 0.5 \gg \sigma_y/E$) Almost all of this deformation takes place in the plastic regime. This can be seen numerically. Because deformations are so large, true stress true strain must be used.

$$\epsilon_{total} = ln(1 + \epsilon_{eng}) = ln(1 + \frac{H}{H_0} - 1) = -0.69 \tag{18}$$

$$\epsilon_{elastic} = \frac{-\sigma_y}{E} = -0.0025 \tag{19}$$

$$\frac{|\epsilon_{elastic}|}{|\epsilon_{total}|} \ll 1.0 \Rightarrow \epsilon_{total} \sim \epsilon_{plastic} \tag{20}$$

You can see the percentage of total strain in the elastic region is only 0.3%.

Part B: Considering now only analysis in the plastic regime, note the following relationship between the material strength, s and the absolute value of plastic strain, $\bar{\epsilon}^p$, is given to be linear:

$$\frac{ds}{d\bar{\epsilon}^p} = h \tag{21}$$

Integrating that from $\bar{\epsilon}^p = 0$ gives:

$$s(\bar{\epsilon}^p) = s_0 + h\bar{\epsilon}^p \tag{22}$$

s_0 is the stress at the beginning of plastic deformation. Because the loading is monotonic, once yielding begins the following relationship is true (σ is negative in compression)

$$s = \bar{\sigma} = |\sigma| = -\sigma \tag{23}$$

Also realize that in compression, axial strain is negative:

$$\bar{\epsilon}^p = -\epsilon^p \tag{24}$$

From part a) it was determined that $\epsilon \approx \epsilon^p$ Therefore we have

$$-\sigma = s_0 - h\epsilon$$

$$\sigma = -s_0 + h\epsilon$$

Denote the condition where $H/H_0 = 0.5$, $\epsilon = -0.69$ with a subscript c.

$$\sigma_c = -s_0 + h\epsilon_c \Rightarrow \sigma_c = -500 + 2000 \times (-0.69) = -1886 MPa \tag{25}$$

Thus σ must equal σ_c when the desired deformation is achieved. The question asks for largest diameter that can be safely used in the machine for this specific test. We have the

constraint that $\sigma = \sigma_c$. The machine has a load capacity of $-100[kN]$, which means that $|P| \leq 100kN$. For uniaxial compression,

$$\sigma = \frac{P}{A} \tag{26}$$

Where P is the axial load and A is the cross sectional area.

$$|P| = |\sigma_c A| \leq 100[kN] \Rightarrow \sigma_c A \geq -100[kN] \tag{27}$$

Realizing that area changes under plastic deformations and then realizing that the governing principle for area change under plastic deformation is volume conservation, we have:

$$A = A_0 \frac{H}{H_0} = 2A_0 \tag{28}$$

So now we have:

$$2\sigma_c A_0 \geq -100[kN] \Rightarrow A_0 \leq \frac{100[kN]}{-2\sigma_c} \tag{29}$$

Rewrite in terms of the diameter

$$A_0 = \frac{\pi d^2}{4}, \tag{30}$$

we have:

$$d \leq \sqrt{\frac{4 \times 100[kN]}{-2\pi\sigma_c}} = \sqrt{\frac{4 \times 100[kN]}{-2\pi(-1.886E^6[kPa])}} = 5.8[mm] \tag{31}$$

Thus, the maximum diameter we can choose that will still be safe and give the dis- placement needed is $d = 5.8[mm]$.

Problem 4 (20 points)

Part A:

$$\frac{P_{max}}{A} = \frac{1.1[MN]}{25[mm] \times 100[mm]} = 440[MPa] > 350[MPa] \tag{32}$$

The as-received bar can not support the P_{max}. The maximum load it can support without plastic deformation is:

$$P_{elastic} = \sigma_y \times A = 350[\frac{N}{mm^2}] \times 25[mm] \times 100[mm] = 0.875MN \tag{33}$$

Part B:

Under plastic deformation, there is no volume change:

$$\sum_{k=1}^{k=3} d\epsilon_{kk}^p = 0 \tag{34}$$

We know that $d\epsilon_{33}^p = 0$, thus we have:

$$d\epsilon_{11}^{(p)} + d\epsilon_{22}^{(p)} = 0 \Rightarrow d\epsilon_{11}^{(p)} = -d\epsilon_{22}^{(p)} = -\frac{dt}{t} \tag{35}$$

Substitution of $d\epsilon_{11}^{(p)}$ and $d\epsilon_{22}^{(p)}$ into the equation for the equivalent plastic strain increment, we have:

$$d\bar{\epsilon}^p = \sqrt{\frac{2}{3}\sum_{i=1}^{3}\sum_{j=1}^{3} d\epsilon_{ij}^{(p)} d\epsilon_{ij}^{(p)}} = \sqrt{\frac{2}{3}((d\epsilon_{11}^{(p)})^2 + (d\epsilon_{22}^{(p)})^2)} = \frac{2}{\sqrt{3}}\frac{|dt|}{t} \tag{36}$$

Because $dt < 0$, the above equation can be rewritten as:

$$d\bar{\epsilon}^p = -\frac{2}{\sqrt{3}}\frac{dt}{t} \tag{37}$$

Integration of the above equation from $\bar{\epsilon}^p = 0$, we get the expression of the equivalent plastic strain:

$$\bar{\epsilon}^p = -\int_{t_0}^{t} \frac{2}{\sqrt{3}}\frac{dt}{t} = \frac{2}{\sqrt{3}}ln\frac{t_0}{t} \tag{38}$$

Part C:

$$\frac{P_{max}}{tw} = s(t) = \sigma_y[1 + \frac{\frac{2}{\sqrt{3}}ln\frac{t_0}{t}}{c}]^N \Rightarrow P_{max} = tw\sigma_y[1 + \frac{\frac{2}{\sqrt{3}}ln\frac{t_0}{t}}{c}]^N \tag{39}$$

Solving the above equation with matlab (Figure 2) for t, we get the maximum rolling=reduced bar thickness: $t_{max} = 23.57[mm]$
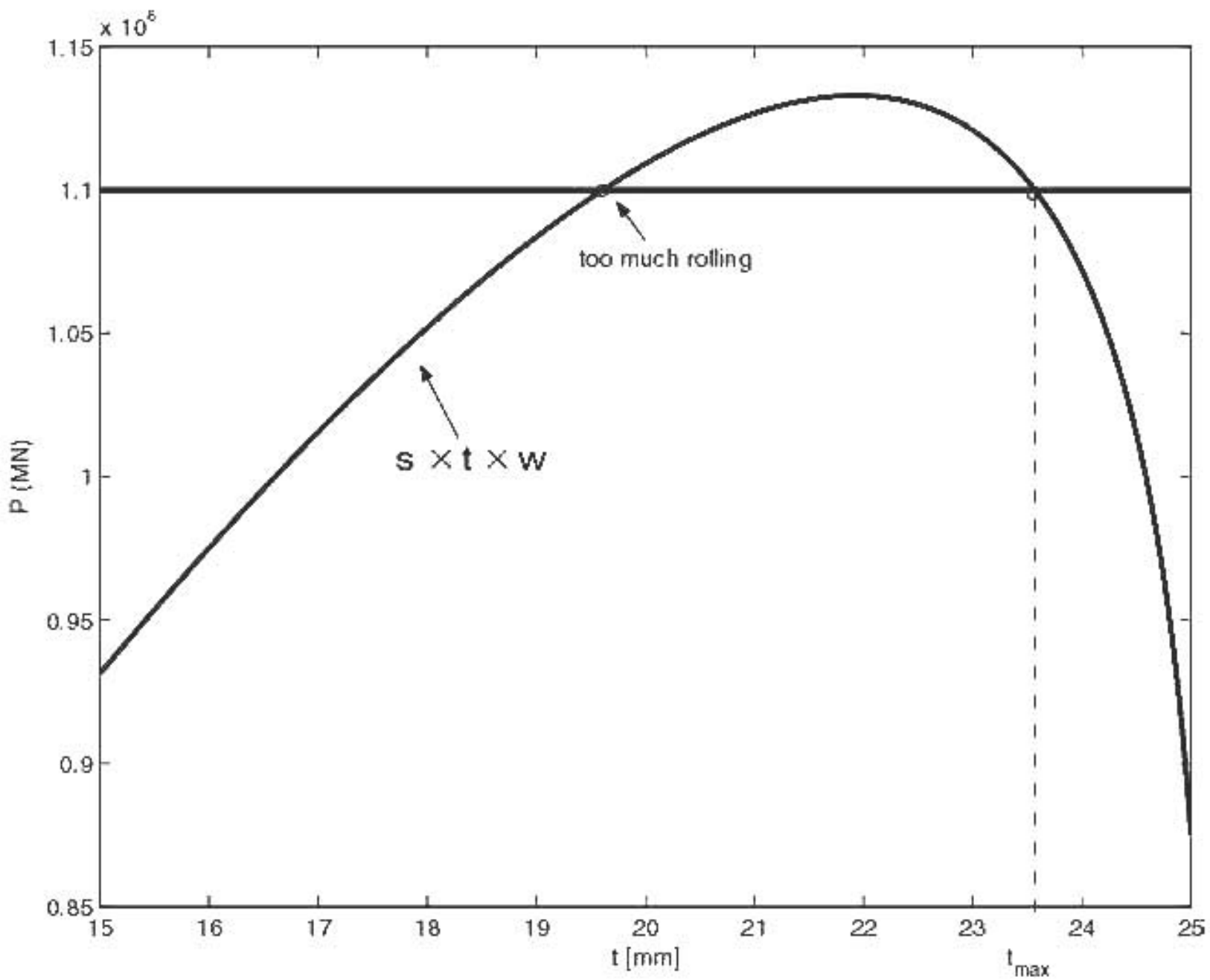
Figure 2