Solutions to Problem Set No. 7

Problem 1. Pendulum mounted on elastic support. This problem is an execise in the application of momentum principles. Two possible solutins are described.

(a) The system consists of two rigid bodies in plane motion. Without constraints there would be six degrees of freedom. However, the carriage m is constrained to slide along the rod, so it has only the single degree of freedom indicated by the displacement x, and the pendulum M is constrained to have its top end connected to the pivot B on the carriage so it has only the single additional degree of freedom indicated by the angle θ.

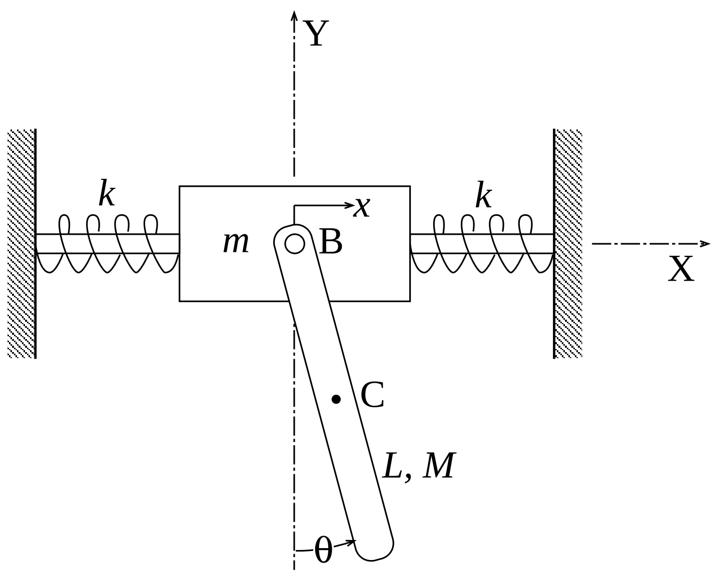

Figure 1: Generalized coordinates x and θ.

The displacements of the mass center C of the pendulum are described with respect to the fixed XY reference frame whose origin is at the equilibrium position of the pivot center B.

$$x_C = x + \frac{L}{2}\sin\theta \qquad \text{and} \qquad y_C = -\frac{L}{2}\cos\theta$$

The linear momentum components of the pendulum are

$$p_x = M\dot{x}_C = M(\dot{x} + \frac{L}{2}\dot{\theta}\cos\theta) \qquad \text{and} \qquad p_y = M\dot{y}_C = M\frac{L}{2}\dot{\theta}\sin\theta$$

The angular momentum of the pendulum about its mass center is

$$H_C = I_c\omega = M\frac{L^2}{12}\dot{\theta}.$$

The constraints on the carriage m prohibit the existence of angular momentum, or linear momentum in the Y-direction. The linear momentum of the carriage in the X-direction is $m\dot{x}$.

(b) Equations of motion for the generalized coordinates x and θ are obtained by applying momentum principles to suitable free-body diagrams of parts of the system. In the first approach, four momentum equations are written which include the two generalized coordinates and two internal reaction forces. The reaction forces are then eliminated by algebraic manipulation to get two simultaneous differential equations for the coordinates x and θ. In the second approach two independent equations of motion for x and θ are obtained directly by careful choice of free bodies and momentum principles. In the second procedure it is necessary to use the generalized angular momentum equation

$$\sum \vec{\tau}_B = \frac{d\vec{H}_B}{dt} + \vec{v}_B \times \vec{P} \tag{1}$$

which applies when the moving moment center B is not the mass center of the system under consideration.

(i) The first approach is more straightforward but may involve considerably more algebraic manipulation. Each mass in the system is isolated in a separate free-body-diagram and as many momentum equations written as there are independent momentum components for that mass. Free-body-diagrams for the carriage (a) and the pendulum (b) are shown in Fig.2. Note that the internal force reaction components N and T at the pivot B in (b) are equal and opposite to those in (a). The external reaction N_1 on the carriage m is applied by the rod which

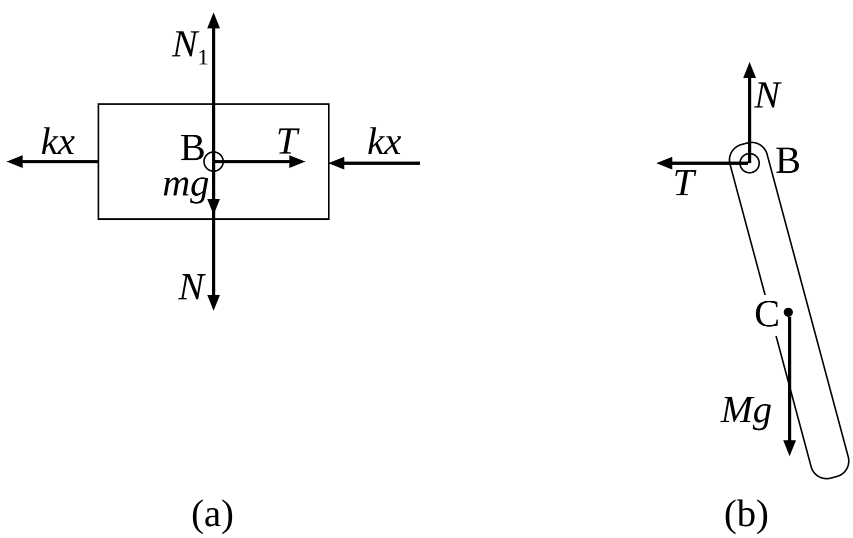

Figure 2: Free-body diagrams.

supports the carriage. Applying the linear momentum principle in the horizontal direction to the free body in (a) yields

$$T - 2kx = \frac{dp_x}{dt} = m\ddot{x} \tag{2}$$

Since the carriage has no vertical momentum the momentum principle in the vertical direction reduces to the equilibrium relation

$$N_1 = mg + N$$

which permits determination of the external reaction force after the internal reaction forces have been found. Turning next to the free-body diagram in (b) of Fig.2 where there are three momentum components: horizontal, vertical, and angular momentum about C, we apply the linear momentum principle in the *horizontal* direction to get

$$-T = \frac{dp_x}{dt} = \frac{d}{dt}M(\dot{x} + \frac{L}{2}\dot{\theta}\cos\theta) = M[\ddot{x} + \frac{L}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)] \qquad (3)$$

and the linear momentum principle in the *vertical* direction to get

$$N - Mg = \frac{dp_y}{dt} = \frac{d}{dt}(M\frac{L}{2}\dot{\theta}\sin\theta) = M\frac{L}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \qquad (4)$$

and the *angular* momentum principle about C to get

$$T\frac{L}{2}\cos\theta - N\frac{L}{2}\sin\theta = \frac{dH_C}{dt} = \frac{ML^2}{12}\ddot{\theta} \qquad (5)$$

At this stage there are four equations [(2), (3), (4), and (5)] for x, θ, N and T. It remains to eliminate the internal reaction forces N and T from these four equations to obtain two independent equations for the generalized coordinates x and θ. One equation is produced by inserting T from (3) into (2) to get

$$(M + m)\ddot{x} + 2kx + M\frac{L}{2}\ddot{\theta}\cos\theta - M\frac{L}{2}\dot{\theta}^2\sin\theta = 0 \qquad (6)$$

A second equation is obtained by inserting the values of T and N provided by (3) and (4) into (5) to get, after considerable cancellation,

$$M\frac{L}{2}\ddot{x}\cos\theta + M\frac{L^2}{3}\ddot{\theta} + Mg\frac{L}{2}\sin\theta = 0 \qquad (7)$$

Equations (6) and (7) are the desired equations of motion for x and θ.

(ii) One way to eliminate the internal force reactions is to consider the *entire system* as a free body. The internal reactions appear as equal and opposite pairs of forces and hence have no net effect on the dynamics of the whole system. A free-body diagram of the entire system is shown in Fig. 3. The equation obtained by

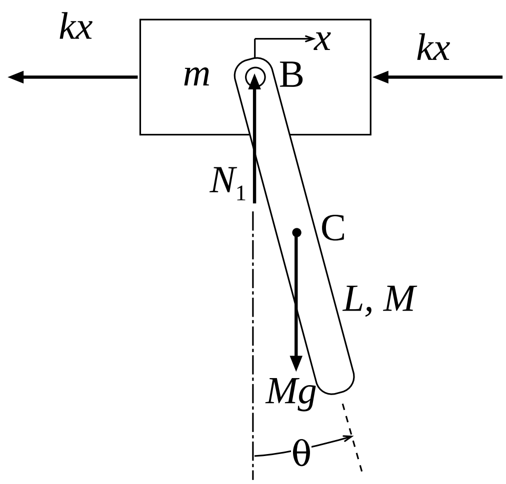

Figure 3: Free-body diagram of carriage plus pendulum.

applying the linear momentum principle in the horizontal direction to the free body in Fig.3 is

$$-2kx = \frac{d}{dt}(m\dot{x}+p_x) = \frac{d}{dt}[m\dot{x}+M(\dot{x}+\frac{L}{2}\dot{\theta}\cos\theta)] = (M+m)\ddot{x}+M\frac{L}{2}(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)$$

which is equivalent to Eq.(6) above.

A second method of eliminating the internal reaction forces N and T is to return to the free-body diagram of the pendulum M in (b) of Fig.2 and observe that these reaction forces have no torque about their point of application B. We therefore consider application of the generalized angular momentum equation (1) to (b) in Fig.2. The only torque about B is due to gravity.

$$\vec{\tau}_B = -Mg\frac{L}{2}\sin\theta\vec{u}_z$$

The angular momentum of the pendulum about B is

$$\begin{aligned}
\vec{H}_B &= I_C\dot{\theta}\vec{u}_z + \overrightarrow{BC}\times\vec{P} = M\frac{L^2}{12}\dot{\theta}\vec{u}_z + \frac{L}{2}(\sin\theta\vec{u}_x - \cos\theta\vec{u}_y)\times(p_x\vec{u}_x + p_y\vec{u}_y) \\
&= M\frac{L^2}{12}\dot{\theta}\vec{u}_z + \frac{L}{2}(\sin\theta\vec{u}_x - \cos\theta\vec{u}_y)\times M[(\dot{x}+\frac{L}{2}\dot{\theta}\cos\theta)\vec{u}_x + (\frac{L}{2}\dot{\theta}\sin\theta)\vec{u}_y)] \\
&= \left[ML^2\dot{\theta}(\frac{1}{12}+\frac{1}{4}\sin^2\theta+\frac{1}{4}\cos^2\theta) + M\frac{L}{2}\dot{x}\cos\theta\right]\vec{u}_z = \left[M\frac{L^2}{3}\dot{\theta} + M\frac{L}{2}\dot{x}\cos\theta\right]\vec{u}_z
\end{aligned}$$

and the additional term in Eq.(1) is

$$\vec{v}_B\times\vec{P} = \dot{x}\vec{u}_x\times(p_x\vec{u}_x + p_y\vec{u}_y) = \dot{x}p_y\vec{u}_z = M\frac{L}{2}\dot{\theta}\dot{x}\sin\theta\vec{u}_z$$

Finally inserting these terms in Eq.(1) yields

$$-Mg\frac{L}{2}\sin\theta = \frac{d}{dt}\left[M\frac{L^2}{3}\dot{\theta} + M\frac{L}{2}\dot{x}\cos\theta\right] + M\frac{L}{2}\dot{\theta}\dot{x}\sin\theta = M\frac{L^2}{3}\ddot{\theta} + M\frac{L}{2}\ddot{x}\cos\theta$$

which is equivalent to Eq.(7) above.

Comparison of procedures (i) and (ii). The first method is straightforward, but the elimination of the superfluous reaction forces can be tedious if the system has many rigid bodies and only a few independent generalized coordinates. The second method requires individual analysis of each new system geometry in order to identify the free bodies and momentum principles which don't introduce any superfluous reaction forces. Much less algebra is required, but every time the generalized angular momentum equation (1) is employed, it is necessary to calculate two vector cross-products: the angular momentum about the moving point B is $\vec{H}_B = \vec{H}_C + \overrightarrow{BC} \times \vec{P}$ and the additional term in (1) is $\vec{v}_B \times \vec{P}$. Depending on the system and on one's experience, these cross-product calculations can be more , or less, time consuming than the algebraic eliminations required in the first procedure.

Problem 2. Stabilization of rocker. The rocker is a single rigid body, but it may be considered to be made up of three parts: two semicicular shapes of mass $m_!$ and a rectangular shape of mass m_2 as shown in Fig.1.

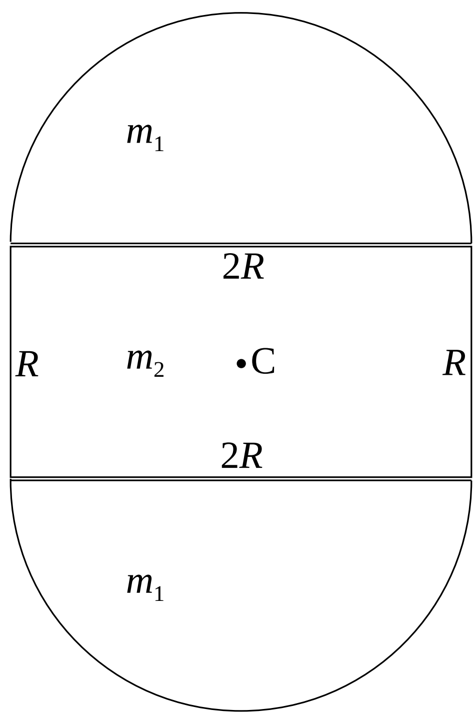

Figure 1: Rocker is made up of three parts.

(a) The mass of the semicircular part is $m_1 = \rho h \pi R^2/2$ and the mass of the rectangular part is $m_2 = 2\rho h R^2$ so the total mass of the rocker is

$$M = 2m_1 + m_2 = \rho h(2 \cdot \pi \frac{R^2}{2} + 2R^2) = \rho h R^2(\pi + 2)$$

Similarly the total moment of inertia of the rocker about its mass center C is

$$I_C = 2I_1 + I_2 \tag{1}$$

where I_1 is the moment of inertia of one of the semicircular parts about C, and I_2 is the moment of inertia of the rectangular part about C. For the rectangular part, the centroid of the rectangle is C itself, so

$$I_2 = m_2 \frac{R^2 + (2R)^2}{12} = \frac{5}{12} m_2 R^2 = \frac{5}{6} \rho h R^4 \tag{2}$$

For the semicircular parts, their centroids C_1 are separated from the rocker centroid C by the distance $R/2 + \overline{y}$ as shown in Fig.2. The calculation of I_1 is an exercise in

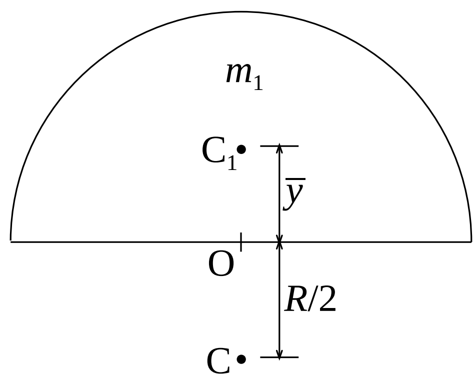

Figure 2: I_1 is the moment of inertia of the semicircular part about C.

the use of the *parallel-axis theorem.* From Fig.2 it follows that

$$I_1 = I_{C_1} + m_1(\frac{R}{2} + \overline{y})^2$$

where I_{C_1} is the moment of inertia of the semicircular part about its own centroid C_1, and $\overline{y}$ is given as $4R/3\pi$. The calculation of I_{C_1} by direct integration is quite tedious. An alternative procedure is to use the parallel-axis theorem again to relate I_{C_1} to the easily evaluated moment of inertia I_o, which is the moment of inertia of the semicicular part about the center O of the circle. The parallel-axis theorem says that

$$I_o = I_{C_1} + m_1 \overline{y}^2 \qquad \text{or} \qquad I_{C_1} = I_o - m_1 \overline{y}^2$$

Now I_o is one half the moment of inertia of a full circular disk of radius R whose mass is $2m_1$, so

$$I_o = \frac{1}{2}(2m_1)\frac{R^2}{2} = m_1 \frac{R^2}{2}$$

Working backwards through the previous equations , one finds

$$I_{C_1} = m_1(\frac{R^2}{2} - \overline{y}^2)$$

and then

$$I_1 = m_1(\frac{R^2}{2} - \overline{y}^2 + \frac{R^2}{4} + R\overline{y} + \overline{y}^2) = m_1 R^2(\frac{3}{4} + \frac{4}{3\pi}) = (\frac{3\pi}{8} + \frac{2}{3})\rho h R^4 \qquad (3)$$

Finally, inserting (2) and (3) into (1) yields

$$I_C = \left[2 \cdot (\frac{3\pi}{8} + \frac{2}{3}) + \frac{5}{6}\right] \rho h R^4 = \frac{9\pi + 26}{12} \rho h R^4 = \frac{9\pi + 26}{12(\pi + 2)} M R^2 = 0.88 M R^2$$

(b) If the rocker is rolled through a small angle away from the equilibrium position and then released from rest the forces acting will be as shown in Fig.3.

It can be seen that the gravity force Mg exerts an *upsetting* torque about the contact

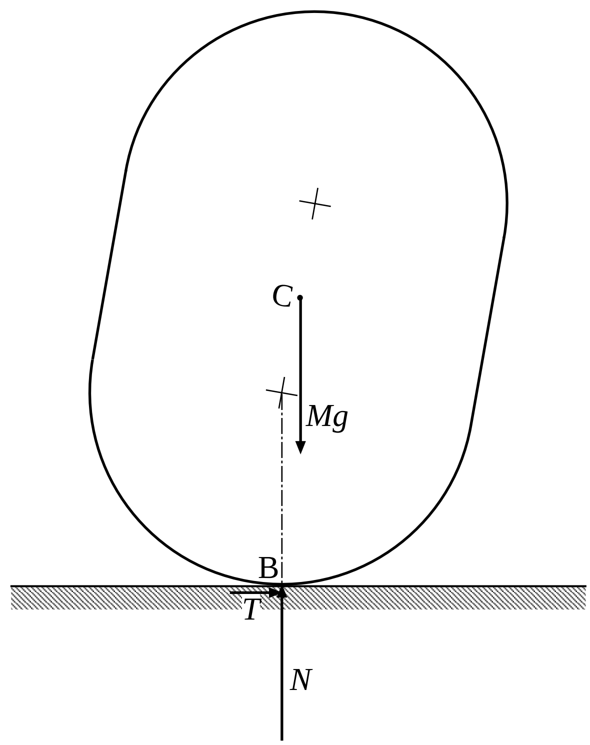

Figure 3: Upright equilibrium position is unstable.

point B. The equilibrium is not stable.

(c) To derive a differential equation which describes the response $\theta(t)$ to the excitation $f(t)$, we begin by studying the rolling motion. See Fig.4. Because of the no-slip

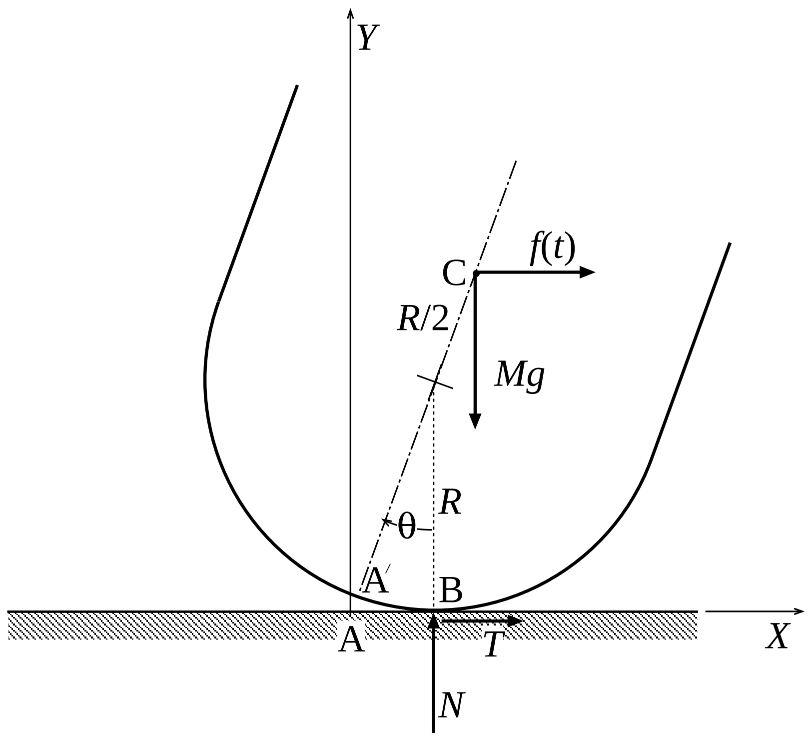

Figure 4: Free-body diagram of rocker rolling without slip.

constraint, only the single generalized coordinate θ is required to completely describe the position of the rocker. Note that the rocker is in the upright equilibrium position when $\theta = 0$. The distance AB rolled along the floor is equal to the arc $\mathrm{A'B} = R\theta$ on the rocker. The displacement components of the mass center C are

$$x = R\theta + \frac{R}{2}\sin\theta \qquad \text{and} \qquad y = R + \frac{R}{2}\cos\theta$$

and the linear momentum components of the rocker are

$$P_x = M\dot{x} = M(R\dot{\theta} + \frac{R}{2}\dot{\theta}\cos\theta) \qquad \text{and} \qquad P_y = M\dot{y} = -M\frac{R}{2}\dot{\theta}\sin\theta$$

The angular momentum of the rocker about its mass center C is

$$H_C = I_C\omega = I_C\dot{\theta}$$

in the clockwise direction. The forces acting on the rocker are shown in Fig.4: The control force $f(t)$ and the gravity force Mg, acting at the mass center C, and the floor reaction force components N and T. Two procedures for obtaining the equation of motion are explained:

(i) In the first procedure three momentum equations are written containing the unknown quantities θ, N, and T, and then the reaction forces are eliminated by algebraic elimination. The three equations represent the linear momentum principle applied in the *horizontal* direction,

$$f(t) + T = \frac{dP_x}{dt} = M(R + \frac{R}{2}\cos\theta)\ddot{\theta} - M\frac{R}{2}\dot{\theta}^2 \sin\theta) \tag{4}$$

the linear mmomentum principle applied in the *vertical* direction,

$$N - Mg = \frac{dP_y}{dt} = -M\frac{R}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \tag{5}$$

and the *angular* momentum principle applied about the mass center C (in the clockwise direction),

$$N\frac{R}{2}\sin\theta - T(R + \frac{R}{2}\cos\theta) = \frac{dH_C}{dt} = I_C\ddot{\theta} \tag{6}$$

It remains to eliminate the floor reaction forces N and T from these three equations. Eq.(4) is easily solved for T, and Eq.(5) is easily solved for N. When these values are inserted in the left-hand side of (6), a single long equation is obtained, which simplifies considerably, on canceling terms proportional to $\sin\theta\cos\theta$, and using the identity $\sin^2\theta + \cos^2\theta = 1$. The simplified result is

$$[I_C + MR^2(\frac{5}{4} + \cos\theta)]\ddot{\theta} - M\frac{R^2}{2}\dot{\theta}^2\sin\theta - Mg\frac{R}{2}\sin\theta = f(t)(R + \frac{R}{2}\cos\theta) \tag{7}$$

This is the desired equation of motion.

(ii) In the second procedure only one momentum principle is applied. From Fig.4, it is seen that the floor reactions N and T will not enter if the angular momentum principle is applied about the contact point B. Since B has the velocity $\vec{v}_B = R\dot{\theta}\vec{u}_x$ it is necessary to use the generalized angular momentum equation

$$\sum \vec{\tau}_B = \frac{d\vec{H}_B}{dt} + \vec{v}_B \times \vec{P} \tag{1}$$

The torque about B is

$$\sum \vec{\tau}_B = -f(t)(R + \frac{R}{2}\cos\theta)\vec{u}_z - Mg\frac{R}{2}\sin\theta\vec{u}_z$$

The angular momentum about B is

$$\vec{H}_B = \vec{H}_C + \overrightarrow{BC} \times \vec{P}$$

where

$$\overrightarrow{BC} = \frac{R}{2}\sin\theta\vec{u}_x + (R + \frac{R}{2}\cos\theta)\vec{u}_y \qquad \text{and} \qquad \vec{P} = P_x\vec{u}_x + P_y\vec{u}_y$$

so

$$\overrightarrow{BC} \times \vec{P} = [P_y\frac{R}{2}\sin\theta - P_x(R + \frac{R}{2}\cos\theta)]\vec{u}z = M\dot{\theta}[-(\frac{R}{2}\sin\theta)^2 - (R + \frac{R}{2}\cos\theta)^2]\vec{u}_z$$

Thus, the angular momentum of the rocker about B is

$$\vec{H}_B = -[I_C\dot{\theta} + MR^2\dot{\theta}(\frac{5}{4} + \cos\theta)]\vec{u}_z$$

and its derivative is

$$\frac{d\vec{H}_B}{dt} = -\{[I_C + MR^2(\frac{5}{4} + \cos\theta)]\ddot{\theta} - MR^2\dot{\theta}^2 \sin\theta\}\vec{u}_z$$

The extra term in Eq.(1) is

$$\vec{v}_B \times \vec{P} = R\dot{\theta}\vec{u}_x \times (P_x\vec{u}_x + P_y\vec{u}_y) = R\dot{\theta}P_y\vec{u}_z = -M\frac{R^2}{2}\dot{\theta}^2 \sin\theta\vec{u}_z$$

Finally, insertion of these terms in Eq.(1) yields

$$-f(t)(R+\frac{R}{2}\cos\theta)\vec{u}_z - Mg\frac{R}{2}\sin\theta = -\{[I_C+MR^2(\frac{5}{4}+\cos\theta)]\ddot{\theta} - MR^2\dot{\theta}^2\sin\theta\} - M\frac{R^2}{2}\dot{\theta}^2\sin\theta$$

which is equivalent to the equation of motion (7) obtained by procedure (i).

$$[I_C + MR^2(\frac{5}{4} + \cos\theta)]\ddot{\theta} - M\frac{R^2}{2}\dot{\theta}^2 \sin\theta - Mg\frac{R}{2}\sin\theta = f(t)(R + \frac{R}{2}\cos\theta) \quad (7)$$

(d) The nonlinear differential equation (7) can be linearized in the neighborhood of the equilibrium position $\theta = 0$ by setting $\sin\theta = \theta$ and $\cos\theta = 1$ and neglecting higher-order terms in θ and $\dot{\theta}$. For example, the term proportional to $\dot{\theta}^2 \sin\theta$ in (7) is of *third* order in comparison with the first-order terms proportional to θ and $\ddot{\theta}$, and is thus omitted from the linear approximation. The linearized approximation to (7) is

$$(I_C + \frac{9}{4}MR^2)\ddot{\theta} - Mg\frac{R}{2}\theta = \frac{3R}{2}f(t) \quad (8)$$

Note that the coefficient of $\ddot{\theta}$ can be interpreted, via the parallel-axis theorem, as the moment of inertia of the rocker about its extreme bottom point A′.

$$I_{A'} = I_C + M(\frac{3R}{2})^2$$

The linearized differential equation in the time domain can be transformed to the Laplace s-domain by making the following sustitutions:

$$\theta(t) \longrightarrow \Theta(s), \qquad f(t) \longrightarrow F(s), \qquad \text{and} \qquad \frac{d}{dt} \longrightarrow s$$

The transform of Eq.(8) is

$$(I_{A'}s^2 - Mg\frac{R}{2})\Theta(s) = \frac{3R}{2}F(s) \quad (9)$$

and the transfer function from $F(s)$ to $\Theta(s)$ is

$$\frac{\Theta(s)}{F(s)} = \frac{\frac{3R}{2}}{I_{A'}s^2 - Mg\frac{R}{2}}$$

(e) The control force $f(t)$ is described in the time domain as

$$f(t) = K[\theta_d(t) - \theta(t)]$$

In the s-domain this becomes

$$F(s) = K[\Theta_d(s) - \Theta(s)] \tag{10}$$

The control force is coupled to the rocker by inserting $F(s)$ from (10) into (9) to get

$$(I_{A'}s^2 - Mg\frac{R}{2})\Theta(s) = K\frac{3R}{2}[\Theta_d(s) - \Theta(s)]$$

or

$$(I_{A'}s^2 + K\frac{3R}{2} - Mg\frac{R}{2})\Theta(s) = K\frac{3R}{2}\Theta_d(s)$$

The transfer function from the desired angle to the actual response angle is

$$\frac{\Theta(s)}{\Theta_d(s)} = \frac{K\frac{3R}{2}}{I_{A'}s^2 + K\frac{3R}{2} - Mg\frac{R}{2}}$$

The poles of the transfer function are the roots of the equation

$$s^2 = -\frac{(3K - Mg)\frac{R}{2}}{I_{A'}}$$

If s^2 is negative the poles lie on the imaginary axis of the s-plane, and the natural motions of the system are bounded oscillations. However, if s^2 is positive, one of the poles is real and positive, indicating that one of the natural motions involves exponential growth. The borderline between stabilty and instability is at $s^2 = 0$, when the gain K has the value

$$K = \frac{Mg}{3}$$

For smaller values of K, the system is unstable; for larger values the system is stable.

Problem 3. Eigenvalue Problem

The equations of motion are obtained by applying the linear momentum principle to each of the masses in turn. The mass $3m$ is acted on by the tension kx_1 to the left, and by the tension $k(x_2 - x_1)$ to the right, so

$$-kx_1 + k(x_2 - x_1) = \frac{d}{dt}(3m\dot{x}_1) = 3m\ddot{x}_1$$

The mass $2m$ is acted on by just the tension $k(x_2 - x_1)$ to the left, so

$$-k(x_2 - x_1) = \frac{d}{dt}(2m\dot{x}_2) = 2m\ddot{x}_2$$

(a) These equations may be written in matrix form as

$$-\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 3m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} \tag{1}$$

(b) The undamped free vibrations are expected to have the form

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \sin(\omega t + \phi)$$

When this trial solution is inserted in (1), the result is the eigenvalue problem

$$\begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \omega^2 \begin{bmatrix} 3m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \tag{2}$$

for the natural mode shapes $\{a_1 \quad a_2\}^T$ and the natural frequencies ω^2.

(c) The analytical solution of the eigenvalue problem is obtained by moving all the terms in (2) to the left-hand side of the equation,

$$\begin{bmatrix} 2k - 3m\omega^2 & -k \\ -k & k - 2m\omega^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3}$$

and setting the determinant of the matrix equal to zero to get the characteristic equation

$$2k^2 - 7km\omega^2 + 6m^2\omega^4 - k^2 = k^2 - 7km\omega^2 + 6m^2\omega^4 = (k - m\omega^2)(k - 6m\omega^2) = 0$$

The roots of the characteristic equation are the eigenvalues

$$\omega_1^2 = \frac{1}{6}\frac{k}{m} \qquad \text{and} \qquad \omega_2^2 = \frac{k}{m}$$

The corresponding natural modes are obtained by back-substituting the eigenvalues in (3). When $\omega_1^2 = \dfrac{1}{6}\dfrac{k}{m}$ is inserted in (3) the result is

$$\begin{bmatrix} \frac{3}{2}k & -k \\ -k & \frac{2}{3}k \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which indicates that the first mode-shape can be represented by the modal vector

$$\{a\}_1 = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_1 = \begin{Bmatrix} 2/3 \\ 1 \end{Bmatrix}$$

When $\omega_2^2 = \dfrac{k}{m}$ is inserted in (3) the result is

$$\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which indicates that the second mode-shape can be represented by the modal vector

$$\{a\}_2 = \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}_2 = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

The modal matrix $[\Phi]$ has the individual modes as columns, so

$$[\Phi] = \begin{bmatrix} 2/3 & -1 \\ 1 & 1 \end{bmatrix}$$

(d) MATLAB does not do dimensions. In order to use MATLAB the eigenvalue problem (2) must be be put in dimensionless form by collecting all the dimensional parameters into a non-dimensional parameter λ to get

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \lambda \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

where

$$\lambda = \frac{m\omega^2}{k}$$

To have MATLAB solve the eigenvalue problem it is only necessary to tell it the non-dimensional [K] and [M] matrices and then type the command [V, D] = eig(K, M). The complete MATLAB session is copied below:

```
K = [ 2 -1 ; -1  1 ] ;  M = [ 3  0 ; 0  2 ] ;
[V, D] = eig( K, M )

V =

    0.7071    0.5547
   -0.7071    0.8321


D =

    1.0000         0
         0    0.1667
```

Note that MATLAB lists the largest eigenvalue first. This is opposite to the usual engineering convention which calls the mode with the *lowest* natural frequency the *first* mode. Note also that MATLAB scales its modal vectors so that the sum of the squares of all the elements is unity. This is different from the common engineering tradition of scaling the modal vector so that the *largest element* is unity.