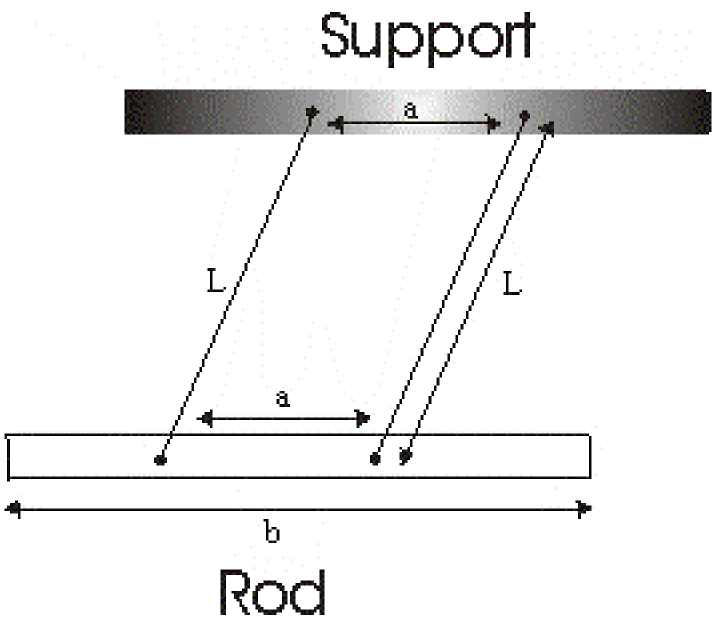
# 2.004: MODELING, DYNAMICS, & CONTROL II

Spring Term 2003

PLEASE ALSO NOTE THAT ALL PRELAB EXERCISES ARE DUE AT THE START (WITHIN 10 MINUTES) OF THE LAB SESSION, NO LATE WORK IS ACCEPTED.

**Pre-Lab Exercise for Experiment 3**

**(1) Symmetric Motion**




(a) Consider a rod pendulum with length b, mass m, suspended by two tethers with length L as shown above. The tethers are separated by distance, a, at the support surface and at the rod. Write down the equation of motion for the center of mass of the rod and deduce its natural oscillation frequency in the absence of loss mechanisms. You may neglect the mass of the tethers.

As long as the tethers are of equal length, equal spacing at top and bottom, and is centered about the center of mass, we can treat this system as all the mass of the rod concentrated at its center of mass on a single tether, or like the problem above.

$$\omega_n = \sqrt{g/L}$$

This can be proven as follows:

$\Sigma F_r = T_{left} + T_{right} - mg\cos\theta;$

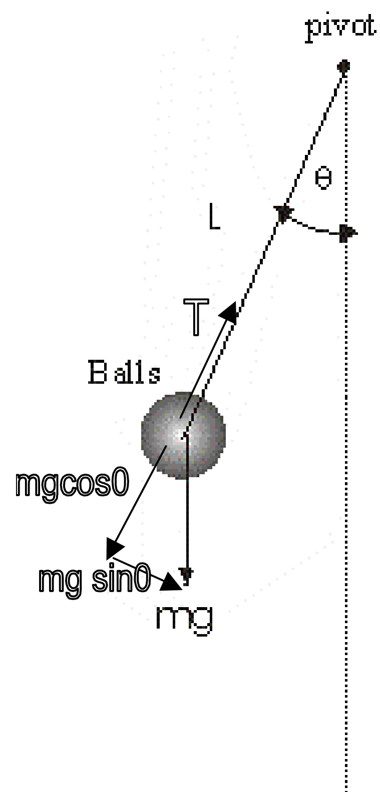
$\Sigma F_\theta = mg\sin\theta$

about the center of mass of the bar:

$\Sigma\tau = -T_{left}*a/2 + T_{right}*a/2 = I*\ddot{\theta}_{CM} = 0$

(no rotation of bar about its CM due to geometric constraints of four-bar mechanism.)

Force in the radial direction:




$-T + mg\cos\theta = ma_r = m(\ddot{L} - L\dot{\theta}^2) = -mL\dot{\theta}^2$,
since $\ddot{L} = 0$

In the tangential direction:

$-mg\sin\theta = ma_\theta = m(2\dot{L}\dot{\theta} + L\ddot{\theta}) = mL\ddot{\theta}$,
since $\dot{L} = 0$

so $L\ddot{\theta} = -g\sin\theta$;

for small $\theta$, $\sin\theta \approx \theta \Rightarrow L\ddot{\theta} = -g\theta$;

Take the Fourier transform and compare to the general second-order equation,

$s^2 + g/L = 0 \quad \Leftrightarrow \quad s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

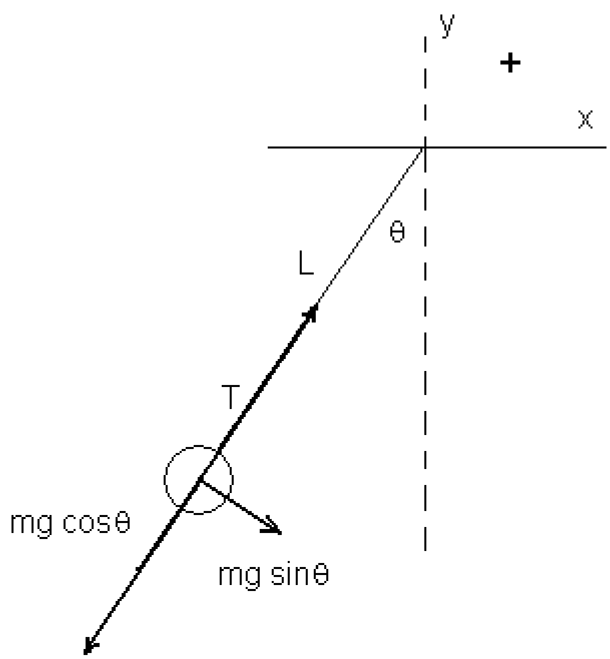
$\Rightarrow \omega_n^2 = g/L$, or $\omega_n = \sqrt{g/L}$

Can also look at this problem from the torque about the pivot point:

$-mg\sin\theta * L = I*\ddot{\theta}$ where $I$ of a point mass at distance $I = m*L^2$

then $\ddot{\theta} + g(\theta/L) = 0$ again.

(b) Write an equation describing the trajectory of the center of mass as a function of time.




$\theta(t) = \theta_{max} \cos(\omega_n \mathrm{t} + \phi)$ and we know that $\omega_n = \sqrt{g/L}$ from before, so

$\theta(t) = \theta_{max} \cos(\sqrt{g/L}\, t + \phi)$

$x(\mathrm{t}) = L \sin\theta(t)$
$y(\mathrm{t}) = -L \cos\theta(t)$

now you have $x$ and $y$ as a function of time.

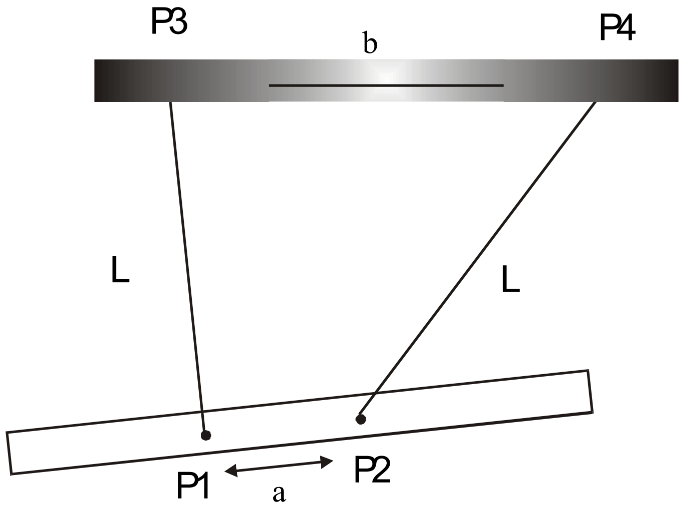
(c) From the solution in part (b), write an equation describing the trajectories of one end of the rod (please pick any one of the ends).

I picked the left end:

$x(t) = L \sin\theta(t) - b/2$
$y(t) = -L \cos\theta(t)$

where $\theta(t)$ is as defined above in part (b)

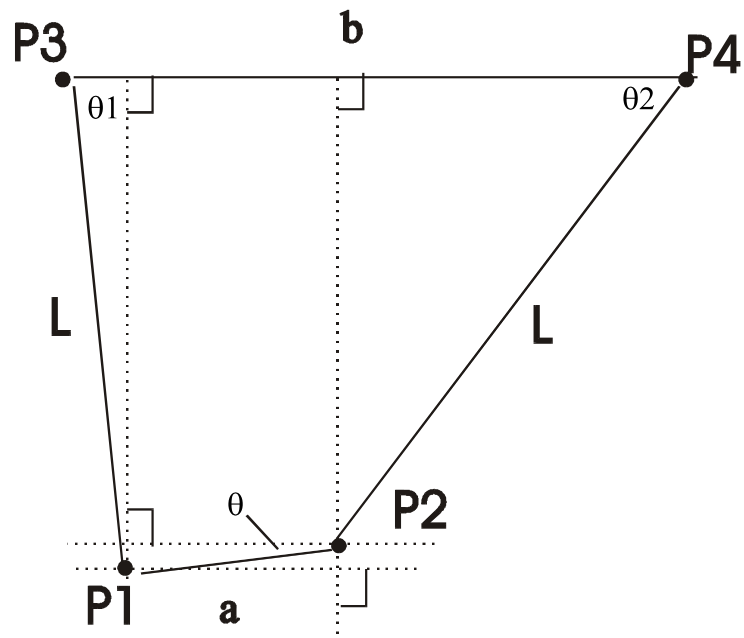
**(2) Asymmetric Pendulum**




(a) Consider a rod pendulum with mass *m*, suspended by two tethers with length *L* as shown above. The tethers are separated by a distance *b* at the support surface and separated by distance *a* at the rod. You may neglect the mass of the tethers. Explain why this system has only one degree of freedom.

This planar system has three degrees of freedom ($x, y, \theta$), but the pin joints constrain motion in the $x$ and $y$ directions. This leaves only $\theta$ at each joint, and since all four joints are connected in a closed loop, one measured $\theta$ describes the motion of all bodies in the system.

(b) Since this system has only one degree of freedom, its kinematics can be described by one generalized coordinate. We can simplify the representation of the geometry of the system at a particular configuration as follows:




Choose $\theta_1$ as the generalized coordinate, express the relationship between $\theta$, $\theta_1$, $\theta_2$.

$$L\cos\theta_1 + a\cos\theta + L\cos\theta_2 = b;$$

and

$$L\sin\theta_1 - a\sin\theta = L\sin\theta_2;$$

(c) At resting configuration, the values of $\theta_1$ and $\theta_2$ are equal by symmetry (why?) and we will designate the equilibrium values of the angles $\theta_1$ and $\theta_2$ as $\theta_0$. Express $\theta_0$ your result in terms of $a$, $b$, and $L$. Redraw the figure at equilibrium.

At rest, $\theta_1$ is equal to $\theta_2$ because it is assumed that distances $a$ and $b$ are centered about the center of mass of the rod, $\therefore$ the bar is parallel to the suspension surface when at rest, so that there is no net torque on the bar.

From part (b),
$L \sin\theta_1 - a \sin\theta = L \sin\theta_2$

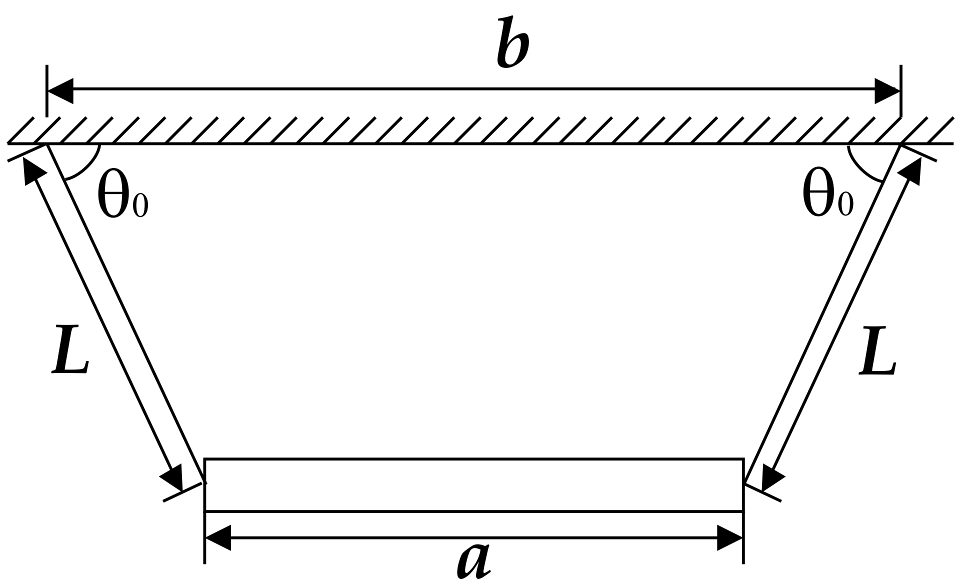
$\theta = 0$ because $\theta_1 = \theta_2 = \theta_o$

$L \cos\theta_1 + a \cos\theta + L \cos\theta_2 = b$

$L \cos\theta_o + a + L \cos\theta_o = b$

$L \cos\theta_o = b/2 - a/2;$

$\theta_o = \cos^{-1}(\frac{b-a}{2L});$



(d) Consider small perturbation of this pendulum from its equilibrium position such that:

$$\theta_1 = \theta_0 + \Delta\theta_1$$

$$\theta_2 = \theta_0 - \Delta\theta_2$$

where $\Delta\theta_1$ , $\Delta\theta_2$ are assumed to be small. Further, you are given the Taylor expansions:

$$\sin(\theta_0 \pm \Delta\theta_i) = \sin\theta_0 \pm \Delta\theta_i \cos\theta_0$$
$$\cos(\theta_0 \pm \Delta\theta_i) = \cos\theta_0 \mp \Delta\theta_i \sin\theta_0$$

Verify that $\Delta\theta_1$ and $\Delta\theta_2$ are equal?

In addition, we should define $\theta$ first.

$\theta = 0 + \Delta\theta = \Delta\theta$

From

$L \cos\theta_1 + a \cos\theta + L \cos\theta_2 = b$

$L \cos(\theta_o + \Delta\theta_1) + a \cos\theta + L \cos(\theta_o - \Delta\theta_2) = b$

$L (\cos\theta_o - \Delta\theta_1 \sin\theta_o) + a \cos\Delta\theta + L (\cos\theta_o + \Delta\theta_2 \sin\theta_o) = b$

$2L \cos\theta_o + a + L \sin\theta_o (\Delta\theta_2 - \Delta\theta_1) = b$

but $2L \cos\theta_o + a = b$ by symmetry, as shown in part c)

$L \sin\theta_o (\Delta\theta_2 - \Delta\theta_1) = 0$

$\Delta\theta_2 - \Delta\theta_1 = 0$

$\Delta\theta_2 = \Delta\theta_1$

ta-da.

Additionally,

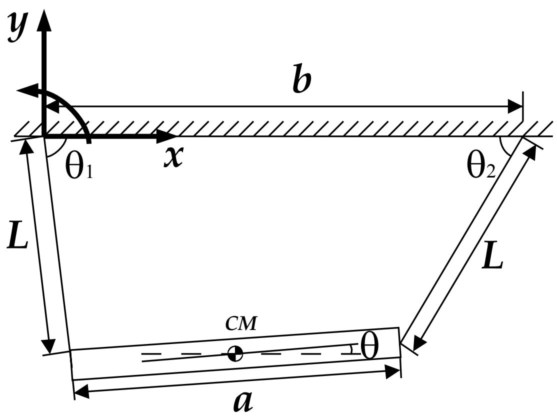
$L \sin\theta_1 - a \sin\theta = L \sin\theta_2$ ;

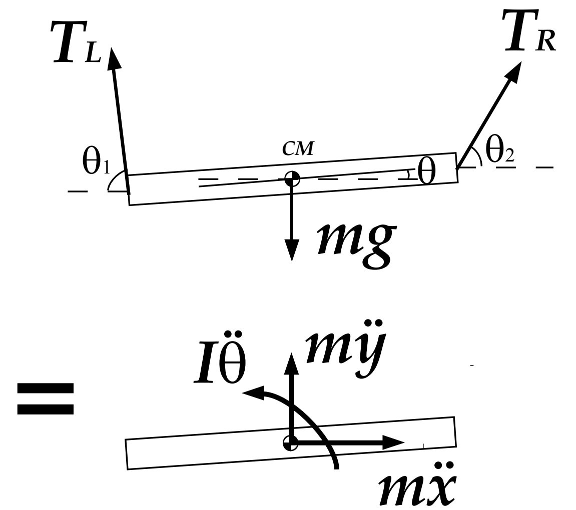
$L \sin(\theta_o + \Delta\theta_1) - a \sin\Delta\theta = L \sin(\theta_o - \Delta\theta_2)$ ;

$L (\sin\theta_o + \Delta\theta_1 \cos\theta_o) - a\Delta\theta = L (\sin\theta_o - \Delta\theta_2 \cos\theta_o)$ ;

$\Delta\theta = (2L/a)\cos\theta_o\Delta\theta_1$ since $\Delta\theta_2 = \Delta\theta_1$

(e) In reference to the center of mass, write the force and moment balance equations in terms of $\Delta\theta_1$.




For small motion,

$\theta_1 = \theta_0 + \Delta\theta_1$

$\theta_2 = \theta_0 - \Delta\theta_2$

$\theta = \Delta\theta$

From the force balance equations,

$\Sigma F_x$: $-T_1 \cos\theta_1 + T_2 \cos\theta_2 = ma_x$;

$\Sigma F_y$: $T_1 \sin\theta_1 + T_2 \sin\theta_2 - mg = ma_y$;

$\Sigma M_{CM}$: $-T_1(a/2)\sin(\theta_1 + \theta) + T_2(a/2)\sin(\theta_2 - \theta) = I\ddot{\theta}$

By the geometry,

$x = L\cos\theta_1 + a/2\cos\theta$;

$y = -L\sin\theta_1 + a/2\sin\theta$

$a_x = \ddot{x} = -L(\cos\theta_1\,\dot{\theta}_1^2 + \sin\theta_1\,\ddot{\theta}_1) - a/2\,(\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta})$,

$a_y = \ddot{y} = L(\sin\theta_1\,\dot{\theta}_1^2 - \cos\theta_1\,\ddot{\theta}_1) - a/2\,(\sin\theta\,\dot{\theta}^2 - \cos\theta\,\ddot{\theta})$

At the geometrical constraint in (b),

$L\sin\theta_1 - a\sin\theta = L\sin\theta_2$,

$L\sin(\theta_0 + \Delta\theta_1) - a\sin\Delta\theta = L\sin(\theta_0 - \Delta\theta_2)$,

$L(\sin\theta_0 + \Delta\theta_1\cos\theta_0) - a\Delta\theta = L(\sin\theta_0 - \Delta\theta_2\cos\theta_0)$,

$L(\Delta\theta_1 + \Delta\theta_2)\cos\theta_0 = a\Delta\theta$,

$\Delta\theta = 2(L/a)\cos\theta_0\Delta\theta_1$

Force balance equations are expressed in terms of $\Delta\theta_1$ by replacing $\Delta\theta$ with $\Delta\theta_1$.

(f) Simply if the equations of motion as a function of $\Delta\theta_1$ . Express the natural frequency of this pendulum in terms of the gravitation constant g, and geometric constants a, b, and L.

From g),

$$T_1 = m\frac{-a_x \sin\theta_2 + (a_y + g)\cos\theta_2}{\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1}$$

$$T_2 = m\frac{a_x \sin\theta_1 + (a_y + g)\cos\theta_1}{\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1}$$

$$\frac{a_x \sin\theta_2 - (a_y + g)\cos\theta_2}{\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1}\sin(\theta_1 + \theta) + \frac{a_x \sin\theta_1 + (a_y + g)\cos\theta_1}{\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1}\sin(\theta_2 - \theta) = \frac{2I}{ma}\ddot{\theta}$$

Based on the small angle approximations

$$a_x \sin\theta_2 \sin(\theta_1 + \theta) \approx -L\sin^3\theta_0 \Delta\ddot{\theta}_1$$
$$a_x \sin\theta_1 \sin(\theta_2 - \theta) \approx -L\sin^3\theta_0 \Delta\ddot{\theta}_1$$
$$a_y \cos\theta_2 \sin(\theta_1 + \theta) \approx -L\cos^2\theta_0 \Delta\ddot{\theta}_1 + \frac{a}{2}\sin\theta_0 \cos\theta_0 \Delta\ddot{\theta}$$
$$a_y \cos\theta_1 \sin(\theta_2 - \theta) \approx -L\cos^2\theta_0 \Delta\ddot{\theta}_1 + \frac{a}{2}\sin\theta_0 \cos\theta_0 \Delta\ddot{\theta}$$

$$g\cos\theta_2 \sin(\theta_1 + \theta) \approx g\left[(\Delta\theta_1 + \Delta\theta)\cos^2\theta_0 + \Delta\theta_2 \sin^2\theta_0\right] = g\left[\Delta\theta_1 + \Delta\theta\cos^2\theta_0\right]$$
$$g\cos\theta_1 \sin(\theta_2 - \theta) \approx g\left[-(\Delta\theta_2 + \Delta\theta)\cos^2\theta_0 - \Delta\theta_1 \sin^2\theta_0\right] = -g\left[\Delta\theta_1 + \Delta\theta\cos^2\theta_0\right]$$

$$\ddot{\theta}(\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1) \approx 2\sin\theta_0 \cos\theta_0 \Delta\ddot{\theta}$$

Plug-in the above approximations into the moment balance equation,

$$-2L\sin^3\theta_0 \Delta\ddot{\theta}_1 - 2g(\Delta\theta_1 + \Delta\theta\cos^2\theta_0) = \frac{4I}{ma}\sin\theta_0 \cos\theta_0 \Delta\ddot{\theta}$$

By using the results in (d) in order to eliminate $\Delta\ddot{\theta}$,

$$\left(L\sin^3\theta_0 + \frac{4IL}{ma^2}\sin\theta_0 \cos^2\theta_0\right)\Delta\ddot{\theta}_1 + g\left(1 + \frac{2L}{a}\cos^3\theta_0\right)\Delta\theta_1 = 0$$

$$L\sin\theta_0\left(ma^2\sin^2\theta_0 + 4I\cos^2\theta_0\right)\Delta\ddot{\theta}_1 + gma\left(a + 2L\cos^3\theta_0\right)\Delta\theta_1 = 0$$

Therefore, the natural frequency for the rod pendulum is

$$\omega_n = \sqrt{\frac{g}{L}\frac{ma\left(a + 2L\cos^3\theta_0\right)}{\sin\theta_0\left(ma^2\sin^2\theta_0 + 4I\cos^2\theta_0\right)}}$$

where

$$\cos\theta_o = \left(\frac{b-a}{2L}\right);$$

$$\sin\theta_o = \left(\frac{\sqrt{4L^2 - (b-a)^2}}{2L}\right), \text{ and}$$

$I = ma^2/12$

$$\rightarrow \qquad \omega_n = \sqrt{\frac{g}{L}\frac{3L\left(4L^2a + (b-a)^3\right)}{a\sqrt{4L^2-(b-a)^2}\left(6L^2-(b-a)^2\right)}}$$

yay.

NOTE: In the experiment, moment of inertia for the rod pendulum is $I = ma_0^2/12$, not $I = ma^2/12$ where $a_0$ is length of rod pendulum. (Actually, it has fixed length in the experiment.) Therefore, the natural frequency for rod pendulum in the experiment will be as below:

$$\omega_n = \sqrt{\frac{g}{L}\frac{6La\left(4L^2a + (b-a)^3\right)}{\sqrt{4L^2-(b-a)^2}\left(12L^2a^2 - \left(3a^2 - a_0^2\right)(b-a)^2\right)}}$$