2.016 Hydrodynamics

Professor A.H. Techet

| **Stream Functions for planar flow** (satisfy $\nabla \cdot \vec{u} = 0$)			
Planar flow: Cartesian $(x, y, \cancel{z})$	$u_x = \frac{\partial \psi}{\partial y}$	$u_y = -\frac{\partial \psi}{\partial x}$	$u_z = 0$
Planar flow: Cylindrical $(r, \theta, \cancel{z})$	$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$	$u_\theta = -\frac{\partial \psi}{\partial r}$	$u_z = 0$
Axisymmetric flow: Cylindrical $(r, \cancel{\theta}, z)$	$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z}$	$u_\theta = 0$	$u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$
Axisymmetric flow: Spherical $(r, \theta, \cancel{\phi})$	$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}$	$u_\theta = -\frac{1}{r \sin\theta}\frac{\partial \psi}{\partial r}$	$u_\varphi = 0$

Potential Functions ($\vec{u} = \nabla\phi$, requires $\nabla \times \vec{u} = 0, \nabla^2\phi = 0$)			
Cartesian coordinates (x, y, z)	$u_x = \frac{\partial \phi}{\partial x}$	$u_y = \frac{\partial \phi}{\partial y}$	$u_z = \frac{\partial \phi}{\partial z}$
Cylindrical coordinates (r, θ, z)	$u_r = \frac{\partial \phi}{\partial r}$	$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$	$u_z = \frac{\partial \phi}{\partial z}$
Spherical coordinates (r, θ, φ)	$u_r = \frac{\partial \phi}{\partial r}$	$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$	$u_\varphi = \frac{1}{r \sin\theta}\frac{\partial \phi}{\partial \varphi}$
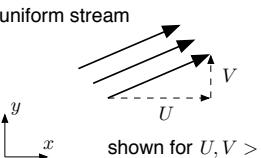
uniform stream shown for $U, V > 0$	$\Phi(z) = (U - iV)z$ <hr> $\phi = Ux + Vy$ $\psi = -Vx + Uy$	$u_x = U$ $u_y = V$
source (Q>0) or sink (Q<0) shown for $Q > 0$	$\Phi(z) = \frac{Q}{2\pi}\ln(z - z_0)$ <hr> $\phi = \frac{Q}{2\pi}\ln r'$ $\psi = \frac{Q}{2\pi}\theta'$	$u_r = \frac{Q}{2\pi}\frac{1}{r'}$ $u_\theta = 0$
free vortex shown for $\Gamma > 0$	$\Phi(z) = \frac{-i\Gamma}{2\pi}\ln(z - z_0)$ <hr> $\phi = \frac{\Gamma}{2\pi}\theta'$ $\psi = -\frac{\Gamma}{2\pi}\ln r'$	$u_r = 0$ $u_\theta = \frac{\Gamma}{2\pi}\frac{1}{r'}$
forced vortex shown for $K > 0$	$\Phi(z) = \nexists$ <hr> $\phi = \nexists$ $\psi = -\frac{Kr'^2}{2}$	$u_r = 0$ $u_\theta = Kr'$
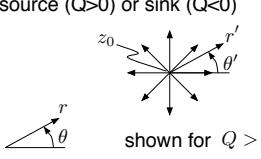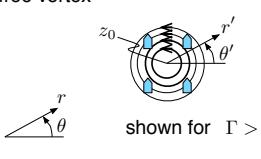
doublet (x-orientation) shown for $c > 0$	$\Phi(z) = \dfrac{c}{z - z_0}$ <hr> $\phi = \dfrac{c \cos \theta'}{r'}$ $\psi = -\dfrac{c \sin \theta'}{r'}$	$u_r = -\dfrac{c \cos \theta'}{r'^2}$ $u_\theta = -\dfrac{c \sin \theta'}{r'^2}$
doublet (y-orientation) shown for $c > 0$	$\Phi(z) = \dfrac{ic}{z - z_0}$ <hr> $\phi = \dfrac{c \sin \theta'}{r'}$ $\psi = \dfrac{c \cos \theta'}{r'}$	$u_r = -\dfrac{c \sin \theta'}{r'^2}$ $u_\theta = \dfrac{c \cos \theta'}{r'^2}$
sphere (axisymmetric flow) shown for $U > 0$	$\Phi(z) = \phi + i\psi$ <hr> $\phi = U \cos \theta' \left(r' + \dfrac{R^3}{2r'^2} \right)$ $\psi = \frac{1}{2} U \sin^2 \theta' \left(r'^2 - \dfrac{R^3}{r'} \right)$	$u_r = U \cos \theta' \left(1 - \dfrac{R^3}{r'^3} \right)$ $u_\theta = -U \sin \theta' \left(1 + \dfrac{R^3}{2r'^3} \right)$ $u_\varphi = 0$
shear flow shown for $A > 0$	$\Phi(z) = \nexists$ <hr> $\phi = \nexists$ $\psi = Ay'^2$	$u_x = 2Ay'$ $u_y = 0$ $u_z = 0$
stagnation point flow shown for $A > 0$	$\Phi(z) = \frac{1}{2} A(z - z_0)^2$ <hr> $\phi = \frac{1}{2} A(x'^2 - y'^2)$ $\psi = Ax'y'$	$u_x = Ax'$ $u_y = -Ay'$ $u_z = 0$
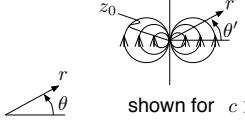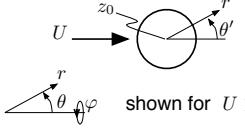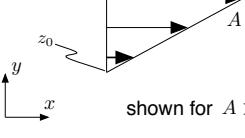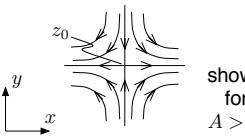
Notes:

$z = x + iy$		$\Phi(z) = \phi + i\psi$
$z_0 = x_0 + iy_0$	$r' = \left[(x - x_0)^2 + (y - y_0)^2 \right]^{\frac{1}{2}}$	$\dfrac{d\Phi}{dz} = u_x - iu_y$
$0 \leq \theta < 2\pi =$	$\theta' = \tan^{-1}\left(\dfrac{y - y_0}{x - x_0} \right)$	$\dfrac{d\Phi}{dz} = (u_r - iu_\theta)e^{-i\theta}$
$u_x = u_r \cos \theta - u_\theta \sin \theta$ $u_y = u_r \sin \theta + u_\theta \cos \theta$	$u_r = u_x \cos \theta + u_y \sin \theta$ $u_\theta = -u_x \sin \theta + u_y \cos \theta$	