Session 16

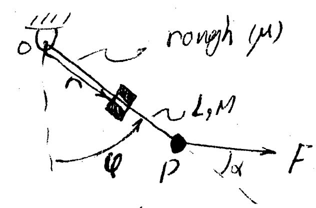


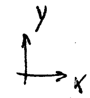
Analytic Mechanics
- generalized Coordinate
- Constraint
- Virtual disp
- Virtual work

$$\delta W = \underline{F}\cdot d\underline{r}$$

- Generalized Force

Example $g\downarrow$




point O is an ideal Constraint and by definition it doesn't do work.

$$\delta W = \underbrace{\delta W^{pot}}_{-\delta V} + \delta W^{non\text{-}pot}$$

$$\hookrightarrow \; Q_r^{pot} = mg\cos\varphi$$

$$Q_\varphi^{pot} = -(M\tfrac{L}{2} + mr)\, g\sin\varphi$$

$$\delta W^{non\text{-}pot} = \delta W_F + \delta W_{friction}$$

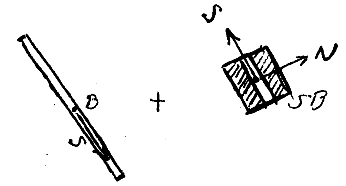
$$\delta W_F = \underline{F}\cdot\delta\underline{r}_P = F(\sin(\varphi+\alpha)\hat{i} - \cos(\varphi+\alpha)\hat{j}) \cdot \delta(L\sin\varphi\,\hat{i} - L\cos\varphi\,\hat{j})$$

$$= FL\sin\alpha\,\delta\varphi$$

$$\delta W_{friction} = \underset{0}{\delta W^{beam}_{friction}} + \underbrace{\delta W^{Collar}_{friction}}_{S\,\delta r}$$

virtual displacement is zero

FBD




$$S = -\mu|N|\,\mathrm{Sign}(\dot{r})$$

what's $\underline{N}$?

Linear momentum principle Applied to Collar $\quad \dot{\underline{P}} = m\ddot{\underline{r}}_B = \underline{N} + m\underline{g} + \underline{S}$

$$(m\ddot{\underline{r}}_B)\cdot\underline{e}_N = N - mg\sin\varphi$$

$$\underline{r}_B = r\sin\varphi\,\hat{i} - r\cos\varphi\,\hat{j}$$

$$\ddot{r}_B = (\ddot{r}\sin\varphi + 2\dot{r}\dot{\varphi}\cos\varphi + r\ddot{\varphi}\cos\varphi + r\dot{\varphi}^2\sin\varphi)\,\hat{i}$$
$$- (\ddot{r}\cos\varphi - 2\dot{r}\dot{\varphi}\sin\varphi - r\ddot{\varphi}\sin\varphi - r\dot{\varphi}^2\cos\varphi)\,\hat{j}$$

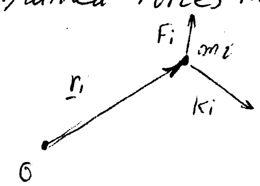
$$a_B\cdot\underline{e}_N = 2\dot{r}\dot{\varphi} + r\ddot{\varphi} \;\Rightarrow\; N = m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + g\sin\varphi) \;\Rightarrow\; S = -\mu m|r\ddot{\varphi} + 2\dot{r}\dot{\varphi} + g\sin\varphi|\,\mathrm{sign}$$

$$\delta W = \overbrace{\left[-g\sin\varphi\left(M\tfrac{L}{2}+mr\right)+F l\sin\alpha\right]}^{Q_\varphi}\delta\varphi$$
$$+\underbrace{\left[mg\cos\varphi-\mu M/\ \ldots /\ \text{sign}(\dot r)\right]}_{Q_r}\delta r$$

Now we turn to deriving eqn of motion using all these ingredients.

First let's formulate an extremum principle for the motion of a mechanical System

Consider a System of $n$ particle $(m_1,\ldots,m_n)$ Subject to active forces $F_i$ and Constrained forces $K_i$




assume Constraints are holonomic
even without this assumption we have

LMP: $\dot{P}_i = F_i + K_i \qquad \left(\sum_{i=1}^{n} \cdot\delta r_i\right)$

$$\Rightarrow \boxed{\sum_{i=1}^{n}(F_i-\dot{P}_i)\cdot\delta r_i = 0} \quad \text{D'Alembert's principle}$$

NOTE: if the System is at equilibrium $(\dot{P}_i = 0) \Rightarrow \boxed{\sum F_i\cdot\delta r_i = 0}$ principle of Virtual work

NOTE: $\displaystyle \cdot \sum_{i=1}^{n} F_i\cdot\delta r_i = \delta W$

$$\cdot \sum_{i=1}^{n}\dot{P}_i\,\delta r_i = \sum_i m_i\ddot{r}_i\cdot\delta r_i = \sum_i m_i\left[\frac{d}{dt}[\dot{r}_i\cdot\delta r_i] - \dot{r}_i\,\delta\dot{r}_i\right]$$
$$= \sum_i m_i\left[\frac{d}{dt}(\dot{r}_i\cdot\delta r_i)\right] - \delta\underbrace{\sum_i \tfrac{1}{2}m_i(\dot{r}_i\cdot\dot{r}_i)}_{T}$$

$$\Rightarrow \delta T + \delta W = \sum_{i=1}^{n} m_i\frac{d}{dt}[\dot{r}_i\cdot\delta r_i]$$

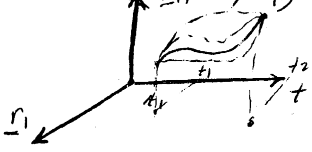
Integrate along the motion $(r_1(t),\ldots,r_n(t+1))$

$$\int_{t_1}^{t_2}\delta(T+W)\Big|_{r(t)}dt = \sum_i\int_{t_1}^{t_2} m_i\frac{d}{dt}[\dot{r}_i\,\delta r_i]\,dt$$

Variation

kinematically admissible path between A & B
Satisfy the Constraints AND Connect A & B
in the extended Configuration Space at $(r_1,\ldots,r_n)$

Geometry



Initial position A: $r_1(t_1), \dots, r_n(t_1)$
Final position B: $r_1(t_2), \dots, r_n(t_2)$

$\Rightarrow \delta r_i \big|_{t=t_1} = 0 \qquad \delta r_i \big|_{t=t_2} = 0$

$$\Rightarrow \int_{t_1}^{t_2} \delta(T+W)\Big|_{r(t)} dt = \sum_i m_i \left[\dot{r}_i \cdot \delta r_i\right]_{t_1}^{t_2} = 0$$

$$\Rightarrow \boxed{\int_{t_1}^{t_2} \delta(T+W)\Big|_{\underline{r}(t)} dt = 0}$$ Extended hamilton Principle (*)

Assume all forces ~~are~~ (active forces) are potential forces

$$\delta W = -\delta V$$

then define the Lagrangian $L = T - V$

$\Rightarrow$ (*) gives $\boxed{\delta \int_{t_1}^{t_2} L(\underline{r}(t), \dot{\underline{r}}(t))\, dt = 0}$ principle of least action (Hamilton's principle)

In other words The function

$$I = \int_{t_1}^{t_2} L(\vec{r}_i(t), \dot{\vec{r}}_i(t), t)\, dt$$

defined for any kinematically admissible path admites an extremum along the actual motion of mechanical system. $\boxed{\delta I = 0}$ $I$ is called the action

Analogy $g(x_1, \dots, x_n)$ [redacted] (function of $n$ variables)

At points of extremay $dg = 0$, indeed $dg = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i = 0$

$$\Leftrightarrow \frac{\partial g}{\partial x_i} = 0 \qquad i = 1, \dots, n$$