Session 18

Analytical Mechanics

For holonomic systems

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = Q_i$$

$$L = T - V; \quad \delta W = \sum Q_i \, \delta q_i$$

Finding Constraint forces using the lagrangian approach

- Consider $q_1, \ldots, q_n$ Complete but not independent set of Coordinate they satisfy Some holonomic Constraint whose Constraint forces we seek
- Assume that we have m holonomic ~~Coordinates~~ Constraints satisfied by these Coordinates

$$\sum_{j=1}^{n} a_{ij} \, dq_j + b_i \, dt = 0 \qquad i = 1, \ldots, m$$

Select Scalars λ_i, $i = 1, \ldots, m$

$$(1) \Rightarrow \sum_{j=1}^{n} a_{ij} \, \delta q_j = 0 \quad \Rightarrow \quad \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{m} a_{ij} \, \delta q_j = 0 \qquad (2)$$

$i = 1, \ldots, m$

By extending hamilton principle

$$(3) \qquad \int_{t_1}^{t_2} (\delta T + \delta V)\Big|_{r(t)} dt = 0$$

Integrate (2) along $r(t)$, add to (3):

$$\int_{t_1}^{t_2} \left(\delta T + \delta W + \sum \lambda_i \, a_{ij} \, \delta q_j\right)\Big|_{r(t)} dt = 0$$

repeat argument leading to lagrang's eqs of motion (except for the last step) to obtain:

$$\sum_{j=1}^{n} \int_{t_1}^{t_2} \left[-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial q_j} + Q_j + \sum_{i=1}^{m} \lambda_i \, a_{ij} \right] \delta q_j \, dt = 0$$

Idea: is making [$]_j$ vanish for all j, use $n-m$ independent δq_j's, AND select $\lambda_1, \ldots, \lambda_m$ in a fashion So that the remaining in brackets vanish

$$\Rightarrow \boxed{\begin{array}{ll} \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} - \dfrac{\partial L}{\partial q_j} = Q_j + \displaystyle\sum_{i=1}^{m} \lambda_i \, a_{ij} & j = 1, \ldots, n \\ \text{Add: } \displaystyle\sum_{j=1}^{n} a_{ij} \, \dot{q}_j + b_i = 0 & i = 1, \ldots, m \end{array}}$$

$n+m$ eqs

$n+m$ unknowns

λ_i: Lagrangian multipliers

$$\sum^{n} a_{ij} \, \dot{q}_j + b_i = 0 \qquad i = 1, \ldots, m$$

NOTE:

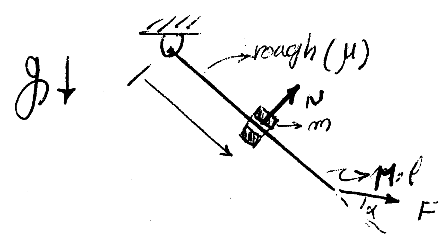
$K_j = \sum_i \lambda_i a_{ij}$

is the j^{th} resultant Coordinate force

Also The above formulation Covers non-holonomic System as well, because Constraints can also be written in the form (1)

Example

Reconsider "Collar sliding on pendulum under the effect of follower force"



Question: Constraint force N ?

. Select: $q_1 = r$, $q_2 = \varphi$ } determine position of center of mass of Collar

$q_3 = \theta$ angle of beam with horizontal

$\Rightarrow n = 3$

. Constraint: $q_2 - q_3 = 0$ $(m = 1)$ $\quad (a_{11} q_1 + a_{12} q_2 + a_{13} q_3 = 0)$

$a_{11} = 0; \quad a_{12} = 1; \quad a_{13} = -1$

. Active generalized forces unrelated to Constraints (non-potential)

in q_1 direction ← $Q_1 = S$

$Q_2 = 0$

$Q_3 = F l \sin\alpha$

, $m = 1 \Rightarrow$ Only one lagrangian multipliers λ_i

$L = T - V$

$T = T_{beam} + T_{collar} = \frac{1}{6} M l^2 \dot\theta^2 + \frac{1}{2} m (\dot r^2 + r^2 \dot\varphi^2)$

$V = V_{beam} + V_{collar} = -Mg\frac{l}{2}\cos\theta - mgr\cos\varphi$.

$$L = \frac{1}{2} M l^2 \dot\theta^2 + \frac{1}{2} m (\dot r^2 + r^2 \dot\varphi^2) + Mg\frac{l}{2}\cos\theta + mgr\cos\varphi$$

Equation of motion:
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = S + \lambda_1 \overset{=0}{a_{11}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = \lambda_1 a_{12} = \lambda_1 \quad K_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = F l \sin\alpha + \lambda_1 a_{13} = F l \sin\alpha - \lambda_1$$

$$K_1 = \lambda_1 = \frac{d}{dt}(m r^2 \dot\varphi) + mgr\sin\varphi$$
$$= 2 m r \dot r \dot\varphi + m r^2 \ddot\varphi + mgr\sin\varphi$$

To obtain N: $\delta W^{non-potential} = (\underbrace{S}_{friction})\delta r + (\underbrace{\overset{Fl\sin\alpha - Nr}{\quad}}_{follower\ force})\delta\theta + \overset{K_j}{Nr}\,\delta\varphi$

$$N = \frac{1}{r} K_2$$
$$= 2m\dot{r}\dot{\varphi} + mr\ddot{\varphi} + mg\sin\varphi$$