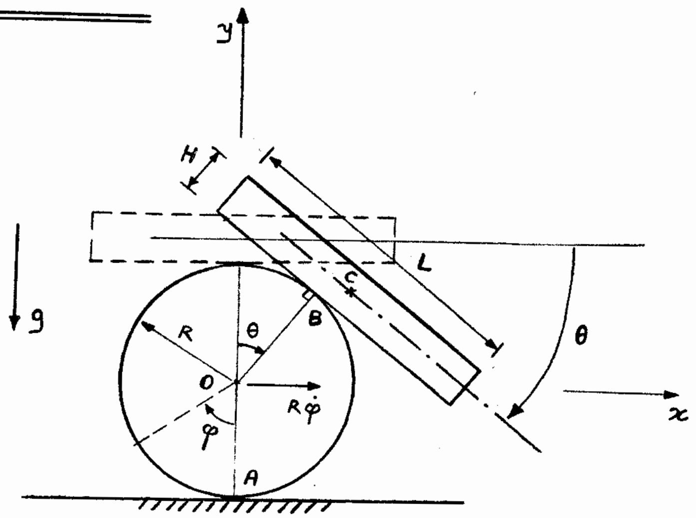
Problem 1

Practice Problems No. 2



(a) xy coordinate system moves with O.

$$\omega|_{cyl.} = -\dot{\varphi}\,\hat{\underset{\sim}{e}}_z$$

$$\omega|_{block} = -\dot{\theta}\,\hat{\underset{\sim}{e}}_z$$

No slip between the cylinder and the floor :

$$\underset{\sim}{v}_0 = R\dot{\varphi}\,\hat{\underset{\sim}{e}}_x$$

$$\underset{\sim}{v}_c = \underset{\sim}{v}_0 + \dot{x}_c\,\hat{\underset{\sim}{e}}_x + \dot{y}_c\,\hat{\underset{\sim}{e}}_y = (R\dot{\varphi} + \dot{x}_c)\,\hat{\underset{\sim}{e}}_x + \dot{y}_c\,\hat{\underset{\sim}{e}}_y$$

No slip between the block and the cylinder:

$$\underset{\sim}{v}_B|_{cyl.} = \underset{\sim}{v}_B|_{block}$$

$$\underset{\sim}{v}_B|_{cyl.} = \underset{\sim}{v}_0 + \omega|_{cyl.} \times \underset{\sim}{OB} = R\dot{\varphi}\,\hat{\underset{\sim}{e}}_x + (-\dot{\varphi}\,\hat{\underset{\sim}{e}}_z) \times (R\sin\theta\,\hat{\underset{\sim}{e}}_x + R\cos\theta\,\hat{\underset{\sim}{e}}_y)$$

$$= R\dot{\varphi}(1+\cos\theta)\,\hat{\underset{\sim}{e}}_x - R\dot{\varphi}\sin\theta\,\hat{\underset{\sim}{e}}_y$$

$$\underset{\sim}{v}_B|_{block} = \underset{\sim}{v}_c + \omega|_{block} \times \underset{\sim}{CB} = (R\dot{\varphi} + \dot{x}_c)\,\hat{\underset{\sim}{e}}_x + \dot{y}_c\,\hat{\underset{\sim}{e}}_y + (-\dot{\theta}\,\hat{\underset{\sim}{e}}_z) \times \left[(R\sin\theta - x_c)\,\hat{\underset{\sim}{e}}_x + (R\cos\theta - y_c)\,\hat{\underset{\sim}{e}}_y\right]$$

$$= (R\dot{\varphi} + \dot{x}_c + R\dot{\theta}\cos\theta - \dot{\theta}y_c)\,\hat{\underset{\sim}{e}}_x + (\dot{y}_c - R\dot{\theta}\sin\theta + \dot{\theta}x_c)\,\hat{\underset{\sim}{e}}_y$$

$$\therefore \begin{cases} R\dot{\varphi}\cos\theta = \dot{x}_c + R\dot{\theta}\cos\theta - \dot{\theta}y_c \\ -R\dot{\varphi}\sin\theta = \dot{y}_c - R\dot{\theta}\sin\theta + \dot{\theta}x_c \end{cases} \Longrightarrow \begin{cases} \sin\theta\,\dot{x}_c + \cos\theta\,\dot{y}_c - \dot{\theta}y_c\sin\theta + \dot{\theta}x_c\cos\theta = 0 \\ R(\dot{\varphi} - \dot{\theta}) = \cos\theta\,\dot{x}_c - \sin\theta\,\dot{y}_c - \dot{\theta}\cos\theta\,y_c - \dot{\theta}x_c\sin\theta \end{cases}$$

$$\Longrightarrow \begin{cases} \dfrac{d}{dt}(\sin\theta\,x_c + \cos\theta\,y_c) = 0 \\ \dfrac{d}{dt}\left[R(\varphi - \theta)\right] = \dfrac{d}{dt}\left[\cos\theta\,x_c - \sin\theta\,y_c\right] \end{cases}$$

Problem 1

(a)

$$\Longrightarrow \begin{cases} \sin\theta\, x_c + \cos\theta\, y_c = \text{Const.} = R + \frac{H}{2} \\ \cos\theta\, x_c - \sin\theta\, y_c = R(\varphi - \theta) \end{cases} \qquad \left(\text{at } \theta = 0,\quad x_c = 0,\quad y_c = R + \frac{H}{2}\right)$$

$$\longrightarrow \begin{cases} x_c = R(\varphi-\theta)\cos\theta + \left(R + \frac{H}{2}\right)\sin\theta \\ y_c = \left(R + \frac{H}{2}\right)\cos\theta - R(\varphi-\theta)\sin\theta \end{cases}$$

(b)

$\xi_1 = \varphi$ & $\xi_2 = \theta$ complete set of independent generalized coordinates

Mass of the block $= m$ $\qquad I_{block} = \frac{1}{12} m (H^2 + L^2)$

$$T = \frac{1}{2} M \underset{\sim}{v}_0 \cdot \underset{\sim}{v}_0 + \frac{1}{2} I_{cyl.}\, \underset{\sim}{\omega}_{cyl.} \cdot \underset{\sim}{\omega}_{cyl.} + \frac{1}{2} m \underset{\sim}{v}_c \cdot \underset{\sim}{v}_c + \frac{1}{2} I_{block}\, \underset{\sim}{\omega}_{block} \cdot \underset{\sim}{\omega}_{block}$$

$$T = \frac{1}{2} M (R\dot\varphi)^2 + \frac{1}{2}\left(\frac{1}{2} M R^2\right)\dot\varphi^2 + \frac{1}{2} m \left[(R\dot\varphi + \dot x_c)^2 + \dot y_c^{\,2} \right] + \frac{1}{2}\left[\frac{1}{12} m (H^2 + L^2)\right]\dot\theta^2$$

$$V = m g y_c + M g \cancelto{0}{y_0} = m g y_c$$

$$T = \frac{3}{4} M R^2 \dot\varphi^2 + \frac{1}{2} m \Big[R^2\dot\varphi^2 + R^2(\varphi-\theta)^2\dot\theta^2 + R^2(\dot\varphi - \dot\theta)^2 + \left(R + \frac{H}{2}\right)^2\dot\theta^2 - 2R^2\dot\theta\dot\varphi(\varphi-\theta)\sin\theta$$
$$+ 2R^2\dot\varphi^2\cos\theta + RH\dot\varphi\dot\theta\cos\theta + 2R(\dot\varphi - \dot\theta)\left(R + \frac{H}{2}\right)\dot\theta \Big] + \frac{1}{24} m (H^2 + L^2)\dot\theta^2$$

$$V = m g \left[\left(R + \frac{H}{2}\right)\cos\theta - R(\varphi - \theta)\sin\theta \right]$$

$$\mathcal{L} = T - V$$

$$\frac{\partial \mathcal{L}}{\partial \dot\varphi} = \frac{3}{2} M R^2 \dot\varphi + \frac{m}{2}\left[2R^2\dot\varphi + 2R^2(\dot\varphi - \dot\theta) - 2R^2(\varphi-\theta)\dot\theta\sin\theta + 2R^2(2\dot\varphi)\cos\theta + RH\dot\theta\cos\theta + 2R\left(R + \frac{H}{2}\right)\dot\theta \right]$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \frac{m}{2}\left[2R^2\dot\theta^2(\varphi - \theta) - 2R^2\dot\theta\dot\varphi\sin\theta \right] + m g R \sin\theta$$

Problem 1

(b)

$$\frac{\partial \mathcal{L}}{\partial \dot\theta} = \frac{m}{2}\Big[2R^2(\varphi-\theta)^2\dot\theta - 2R^2(\dot\varphi-\dot\theta) + 2\left(R+\frac{H}{2}\right)^2\dot\theta - 2R^2\dot\varphi(\varphi-\theta)\sin\theta + RH\dot\varphi\cos\theta$$
$$+ 2R(\dot\varphi-2\dot\theta)\left(R+\frac{H}{2}\right)\Big] + \frac{m}{12}(H^2+L^2)\dot\theta$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{m}{2}\Big[2R^2\dot\theta^2(\theta-\varphi) - 2R^2\dot\theta\dot\varphi\,\varphi\cos\theta + 2R^2\dot\theta\dot\varphi\sin\theta + 2R^2\dot\theta\dot\varphi\,\theta\cos\theta - 2R^2\dot\varphi^2\sin\theta$$
$$-RH\dot\varphi\dot\theta\sin\theta\Big] + mg\left(R+\frac{H}{2}\right)\sin\theta + mgR\varphi\cos\theta - mgR\theta\cos\theta - mgR\sin\theta$$

$$\Xi_\varphi = \Xi_\theta = 0$$

$\delta\varphi$:

$$\left[\frac{3}{2}MR^2 + 2mR^2(1+\cos\theta)\right]\ddot\varphi + \left[-mR^2(\varphi-\theta)\sin\theta + \frac{mRH}{2}(\cos\theta+1)\right]\ddot\theta$$
$$+ \cdots - mgR\sin\theta = 0$$

$\delta\theta$:

$$\left[mR^2(\varphi-\theta)^2 + m\left(\frac{H^2}{3}+\frac{L^2}{12}\right)\right]\ddot\theta + \left[-mR^2(\varphi-\theta)\sin\theta + \frac{mRH}{2}(\cos\theta+1)\right]\ddot\varphi$$
$$+ \cdots - mg\frac{H}{2}\sin\theta + mgR\cos\theta(\theta-\varphi) = 0$$

Governing Equations of Motion

Note that missing terms do not play a role in the following linear stability analysis.

(c)

θ and φ are non-ignorable coordinates. $\Longrightarrow$ equilibrium position $\theta=\theta_s$, $\varphi=\varphi_s$

$$\delta\varphi \Longrightarrow -mgR\sin\theta_s = 0 \longrightarrow \theta_s = 0 \qquad \cancel{\theta_s = \pi}$$

$$\delta\theta \Longrightarrow -mg\frac{H}{2}\sin\theta_s + mgR\cos\theta_s(\theta_s-\varphi_s) = 0 \xrightarrow{\theta_s=0} \varphi_s = 0$$

$\therefore$ Equilibrium position $\underline{\theta_s = \varphi_s = 0}$

Problem 1

(c)

$$\theta = \theta_s + \varepsilon_\theta(t) = \varepsilon_\theta(t)$$

$$\varphi = \varphi_s + \varepsilon_\varphi(t) = \varepsilon_\varphi(t)$$

$$\delta\theta \implies \left(4mR^2 + \frac{3}{2}MR^2\right)\ddot{\varepsilon}_\varphi + mRH\,\ddot{\varepsilon}_\theta - mgR\,\varepsilon_\theta = 0$$

$$\delta\varphi \implies \left(m\frac{H^2}{3} + m\frac{L^2}{12}\right)\ddot{\varepsilon}_\theta + mRH\,\ddot{\varepsilon}_\varphi + mg\left(R - \frac{H}{2}\right)\varepsilon_\theta - mgR\,\varepsilon_\varphi = 0$$

$$\begin{bmatrix} R^2\left(\frac{3}{2}M + 4m\right) & mRH \\ mRH & m\left(\frac{H^2}{3} + \frac{L^2}{12}\right) \end{bmatrix} \begin{Bmatrix} \ddot{\varepsilon}_\varphi \\ \ddot{\varepsilon}_\theta \end{Bmatrix} + \begin{bmatrix} 0 & -mgR \\ -mgR & mg\left(R - \frac{H}{2}\right) \end{bmatrix} \begin{Bmatrix} \varepsilon_\varphi \\ \varepsilon_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\begin{Bmatrix} \varepsilon_\varphi \\ \varepsilon_\theta \end{Bmatrix} = \begin{Bmatrix} x \\ y \end{Bmatrix} e^{i\omega t}$

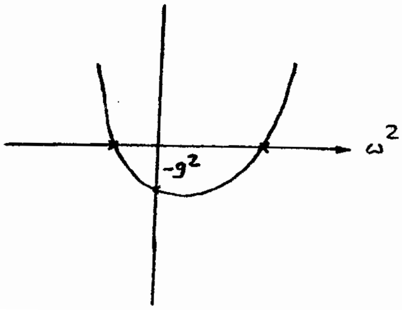
$$\therefore \begin{bmatrix} -\omega^2 R^2\left(\frac{3}{2}M + 4m\right) & -mR\left(g + \omega^2 H\right) \\ -mR\left(g + \omega^2 H\right) & +mg\left(R - \frac{H}{2}\right) - \omega^2 m\left(\frac{H^2}{3} + \frac{L^2}{12}\right) \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Non-trivial solution : $\begin{vmatrix} -\omega^2 R^2\left(\frac{3}{2}M + 4m\right) & -mR\left(g + \omega^2 H\right) \\ -mR\left(g + \omega^2 H\right) & mg\left(R + \frac{H}{2}\right) - \omega^2 m\left(\frac{H^2}{3} + \frac{L^2}{12}\right) \end{vmatrix} = 0$

$$\implies \left[\frac{M}{m}\left(\frac{H^2}{2} + \frac{L^2}{8}\right) + \frac{1}{3}\left(H^2 + L^2\right)\right]\omega^4 + \left[-4Rg + \frac{3}{4}\frac{M}{m}Hg - \frac{3}{2}\frac{M}{m}Rg\right]\omega^2 - g^2 = 0$$

This equation has a negative root for ω^2.

So $\varphi_s = \theta_s = 0$ is <u>unstable</u>.



Problem 1

Note that in the limiting case of $\frac{M}{m} \longrightarrow \infty$, the problem turns into the rolling block on a fixed cylinder. In that case, we get

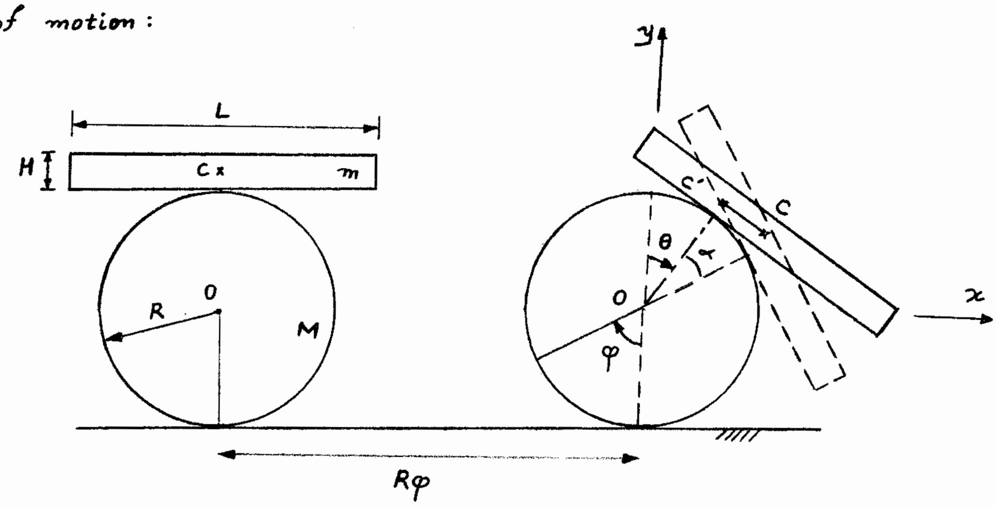
$$\left[\left(\frac{H^2}{2}+\frac{L^2}{8}\right)+\cancel{\frac{1}{3}\frac{m}{M}\left(H^2+L^2\right)}^{\,0}\right]\omega^4+\left[\cancel{-4Rg\frac{m}{M}}^{\,0}+\frac{3}{4}Hg-\frac{3}{2}Rg\right]\omega^2-\cancel{\frac{m}{M}g^2}^{\,0}=0$$

$$\left(\frac{H^2}{2}+\frac{L^2}{8}\right)\omega^2+\frac{3}{4}g\left(H-2R\right)=0$$

$$\longrightarrow \quad \omega^2=\frac{\frac{3}{4}g\left(2R-H\right)}{\frac{H^2}{2}+\frac{L^2}{8}} \quad \longrightarrow \quad \begin{cases} \theta_s=0 \text{ is stable for } \boxed{H<2R} \\ \theta_s=0 \text{ is unstable for } \boxed{H>2R} \end{cases}$$

* Another way to find the location of the center of mass c of the block (x_c, y_c) by studying the geometry of motion:

To analyze the block rolling constraint, assume the block is glued to the cylinder during rotation φ and then rolls back to location θ.

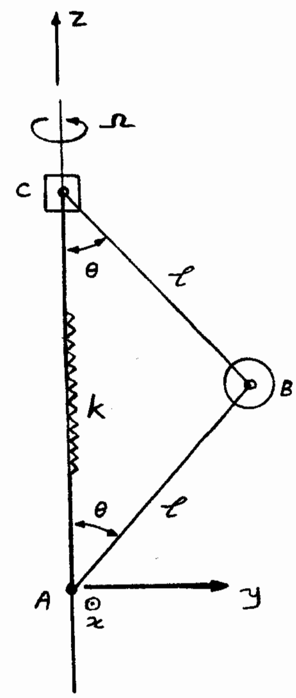


Let roll-back angle be α $(\alpha=\varphi-\theta)$. The absolute rotation of the block is through angle θ.

Location of center of mass of the block: (No slip between the block and the cylinder yields $cc' = R\alpha = R(\varphi-\theta)$)

$$\begin{cases} x_c=\left(R+\frac{H}{2}\right)\sin\theta+R\left(\varphi-\theta\right)\cos\theta \\ y_c=\left(R+\frac{H}{2}\right)\cos\theta-R\left(\varphi-\theta\right)\sin\theta \end{cases}$$

Problem 2



xyz rotates about z with Ω.

Assume small disk B and slider C are point masses.

$$\underset{\sim}{AC} = 2\ell\cos\theta\, \underset{\sim}{\hat{e}}_z$$

$$\underset{\sim}{v}_C = \frac{d}{dt}\underset{\sim}{AC} = -2\ell\sin\theta\dot{\theta}\underset{\sim}{\hat{e}}_z$$

$$\underset{\sim}{v}_B = \frac{d}{dt}\underset{\sim}{AB} = \underset{\sim}{\omega}_{AB}\times\underset{\sim}{AB}$$

$$\underset{\sim}{\omega}_{AB} = -\dot{\theta}\underset{\sim}{\hat{e}}_x + \Omega\underset{\sim}{\hat{e}}_z \qquad \underset{\sim}{AB} = \ell\sin\theta\,\underset{\sim}{\hat{e}}_y + \ell\cos\theta\,\underset{\sim}{\hat{e}}_z$$

$$\underset{\sim}{v}_B = \left(-\dot{\theta}\underset{\sim}{\hat{e}}_x + \Omega\underset{\sim}{\hat{e}}_z\right)\times\left(\ell\sin\theta\,\underset{\sim}{\hat{e}}_y + \ell\cos\theta\,\underset{\sim}{\hat{e}}_z\right) = -\dot{\theta}\ell\sin\theta\,\underset{\sim}{\hat{e}}_z + \dot{\theta}\ell\cos\theta\,\underset{\sim}{\hat{e}}_y - \Omega\ell\sin\theta\,\underset{\sim}{\hat{e}}_x$$

$$T = \frac{1}{2}m\underset{\sim}{v}_B\cdot\underset{\sim}{v}_B + \frac{1}{2}m\underset{\sim}{v}_C\cdot\underset{\sim}{v}_C = \frac{1}{2}m\left(\dot{\theta}^2\ell^2 + \Omega^2\ell^2\sin^2\theta\right) + \frac{1}{2}m\left(4\ell^2\sin^2\theta\,\dot{\theta}^2\right)$$

$$V = mgy_C + mgy_B + \frac{1}{2}k\left(2\ell - 2\ell\cos\theta\right)^2 = 3mg\ell\cos\theta + 2k\ell^2(1-\cos\theta)^2$$

$$\mathcal{L} = T - V$$

$\underline{\underline{\delta\theta}}$:

$$\frac{d}{dt}\left(m\ell^2\dot{\theta} + 4m\ell^2\dot{\theta}\sin^2\theta\right) - m\Omega^2\ell^2\sin\theta\cos\theta - 4m\ell^2\dot{\theta}^2\sin\theta\cos\theta - 3mg\ell\sin\theta + 4k\ell^2(1-\cos\theta)\sin\theta = 0$$

θ is non-ignorable $\Longrightarrow$ steady motion $\theta = \theta_s$

Problem 2

$$\left[-m\Omega^2\ell^2\cos\theta_s - 3mg\ell + 4k\ell^2(1-\cos\theta_s)\right]\sin\theta_s = 0$$

(A) $\sin\theta_s = 0 \quad \longrightarrow \quad \underline{\theta_s = 0} \qquad\qquad \left(0 \leqslant \theta_s < \frac{\pi}{2}\right)$

(B) $\cos\theta_s = \dfrac{4k\ell^2 - 3mg\ell}{4k\ell^2 + m\Omega^2\ell^2} \quad \longrightarrow \quad \underline{\theta_s = \cos^{-1}\left(\dfrac{4k\ell - 3mg}{4k\ell + m\Omega^2\ell}\right)}$

possible only if $0 < \dfrac{4k\ell-3mg}{4k\ell+m\Omega^2\ell} < 1 \quad \Longrightarrow \quad \underline{k > \dfrac{3mg}{4\ell}}$

Stability of θ_s :

Hold Ω constant and supply τ_z as needed.

$\theta = \theta_s + \varepsilon(t)$

$$\delta\theta \quad \Longrightarrow \quad m\ell^2\ddot{\varepsilon} + 4m\ell^2\sin^2\theta_s\,\ddot{\varepsilon} \cancel{- m\Omega^2\ell^2\sin\theta_s\cos\theta_s} - m\Omega^2\ell^2\cos^2\theta_s\,\varepsilon + m\Omega^2\ell^2\sin^2\theta_s\,\varepsilon$$
$$\cancel{-3mg\ell\sin\theta_s} - 3mg\ell\cos\theta_s\,\varepsilon + \cancel{4k\ell^2(1-\cos\theta_s)\sin\theta_s} + 4k\ell^2\sin^2\theta_s\,\varepsilon$$
$$+ 4k\ell^2\cos\theta_s(1-\cos\theta_s)\,\varepsilon = 0$$

$O(1)$ terms cancel. $\quad \left\{\left[-m\Omega^2\ell^2\cos\theta_s - 3mg\ell + 4k\ell^2(1-\cos\theta_s)\right]\sin\theta_s = 0\right\}$

$\longrightarrow$

$$m\ell^2(1+4\sin^2\theta_s)\,\ddot{\varepsilon} + \left[m\Omega^2\ell^2(\sin^2\theta_s - \cos^2\theta_s) - 3mg\ell\cos\theta_s + 4k\ell^2(\sin^2\theta_s - \cos^2\theta_s) + 4\ell^2\cos\theta_s\right]\varepsilon = 0$$

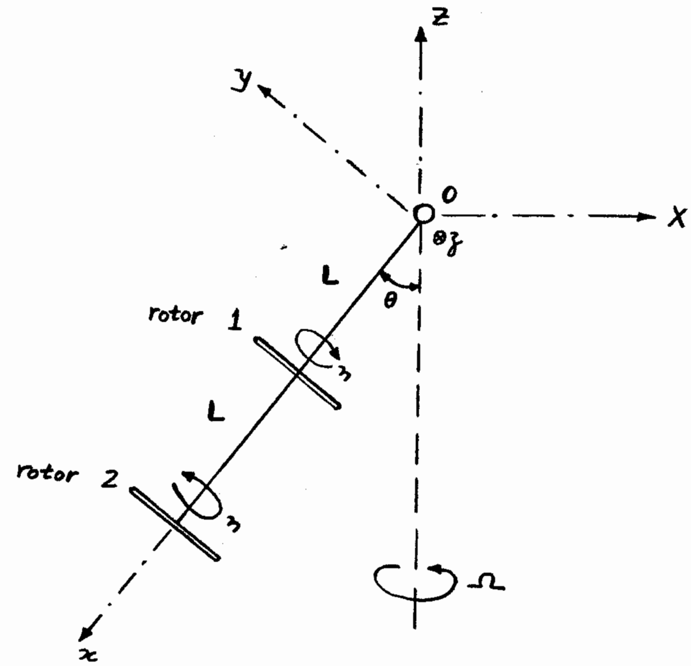
(A) $m\ell^2\ddot{\varepsilon} + (-m\Omega^2\ell^2 - 3mg\ell)\,\varepsilon = 0 \quad \longrightarrow \quad \underline{\theta_s = 0 \text{ is unstable.}}$

(B) $m\ell^2(1+4\sin^2\theta_s)\,\ddot{\varepsilon} + (m\Omega^2\ell^2\sin^2\theta_s + 4k\ell^2\sin^2\theta_s)\,\varepsilon = 0$

$\longrightarrow \quad \ddot{\varepsilon} + \left(\dfrac{\Omega^2 + 4k/m}{1+4\sin^2\theta_s}\right)\sin^2\theta_s\,\varepsilon = 0 \quad \longrightarrow \quad \underline{\theta_s = \cos^{-1}\left(\dfrac{4k\ell-3mg}{4k\ell+m\Omega^2\ell}\right) \text{ is stable.}}$

exists only if $\underline{k > \dfrac{3mg}{4\ell}}$

Problem 3



xyz coordinate system rotates with Ω about Z.

$$\underset{\sim}{\tau}_0 = \frac{d}{dt}\underset{\sim}{H}_0 \qquad (\underset{\sim}{v}_0 = \underset{\sim}{0})$$

$$\underset{\sim}{H}_0 = [I]_0\,\underset{\sim}{\omega}$$

$$\underset{\sim}{H}_0 = [I]_0\big|_1\,\underset{\sim}{\omega}\big|_1 + [I]_0\big|_2\,\underset{\sim}{\omega}\big|_2$$

$$[I]_0\big|_1 = \begin{bmatrix} I_3 & 0 & 0 \\ & I_1 & 0 \\ & & I_1 \end{bmatrix} + m\begin{bmatrix} 0 & 0 & 0 \\ & L^2 & 0 \\ & & L^2 \end{bmatrix} = \begin{bmatrix} I_3 & 0 & 0 \\ & I_1+mL^2 & 0 \\ & & I_1+mL^2 \end{bmatrix}$$

$$[I]_0\big|_2 = \begin{bmatrix} I_3 & 0 & 0 \\ & I_1+4mL^2 & 0 \\ & & I_1+4mL^2 \end{bmatrix}$$

$$\underset{\sim}{\omega}\big|_1 = \dot\psi_1\,\underset{\sim}{\hat e}_x + \Omega\,\underset{\sim}{\hat e}_Z = \dot\psi_1\,\underset{\sim}{\hat e}_x + \Omega\left(-\cos\theta\,\underset{\sim}{\hat e}_x + \sin\theta\,\underset{\sim}{\hat e}_y\right) = \overbrace{\left(\dot\psi_1 - \Omega\cos\theta\right)}^{n}\underset{\sim}{\hat e}_x + \Omega\sin\theta\,\underset{\sim}{\hat e}_y$$

$$\underset{\sim}{\omega}\big|_2 = \dot\psi_2\,\underset{\sim}{\hat e}_x + \Omega\,\underset{\sim}{\hat e}_Z = \dot\psi_2\,\underset{\sim}{\hat e}_x + \Omega\left(-\cos\theta\,\underset{\sim}{\hat e}_x + \sin\theta\,\underset{\sim}{\hat e}_y\right) = \underbrace{\left(\dot\psi_2 - \Omega\cos\theta\right)}_{-n}\underset{\sim}{\hat e}_x + \Omega\sin\theta\,\underset{\sim}{\hat e}_y$$

(Note that, in this problem, n is <u>absolute</u> spin about x)

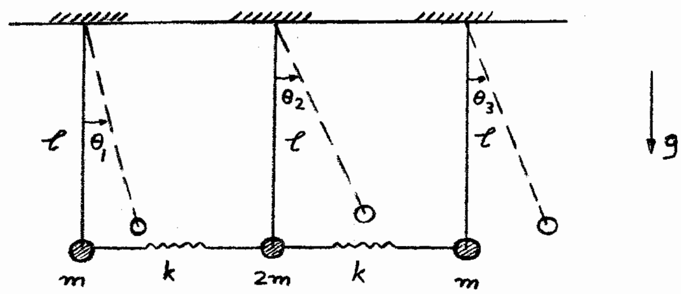
$$\therefore\quad \underset{\sim}{H}_0 = \begin{bmatrix} I_3 & 0 & 0 \\ & I_1+mL^2 & 0 \\ & & I_1+mL^2 \end{bmatrix}\begin{Bmatrix} n \\ -\Omega\sin\theta \\ 0 \end{Bmatrix} + \begin{bmatrix} I_3 & 0 & 0 \\ & I_1+4mL^2 & 0 \\ & & I_1+4mL^2 \end{bmatrix}\begin{Bmatrix} -n \\ -\Omega\sin\theta \\ 0 \end{Bmatrix} = (2I_1+5mL^2)\,\Omega\sin\theta\,\underset{\sim}{\hat e}_y$$

$$\underset{\sim}{\tau}_0 = \frac{d}{dt}\underset{\sim}{H}_0 = \Omega\sin\theta\,(2I_1+5mL^2)\,\frac{d}{dt}\underset{\sim}{\hat e}_y = \boxed{-\Omega^2\sin\theta\cos\theta\,(2I_1+5mL^2)\,\underset{\sim}{\hat e}_z}$$

required torque to maintain the precession

$$\Omega\,\underset{\sim}{\hat e}_Z \times \underset{\sim}{\hat e}_y = -\Omega\cos\theta\,\underset{\sim}{\hat e}_z$$

Problem 4



$$T = \frac{1}{2} m (\ell \dot\theta_1)^2 + \frac{1}{2}(2m)(\ell\dot\theta_2)^2 + \frac{1}{2} m (\ell\dot\theta_3)^2$$

For θ_1, θ_2 and θ_3 small, potential energy correct to quadratic terms in $\theta_1, \theta_2, \theta_3$:

$$V \approx mg\frac{\ell}{2}\theta_1^2 + 2mg\frac{\ell}{2}\theta_2^2 + mg\frac{\ell}{2}\theta_3^2 + \frac{1}{2}k\ell^2(\theta_2-\theta_1)^2 + \frac{1}{2}k\ell^2(\theta_3-\theta_2)^2$$

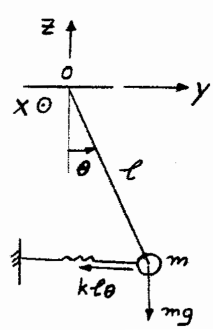
$$\Xi_1 = \Xi_2 = \Xi_3 = 0$$

$$\mathcal{L} = T - V$$

$$\begin{array}{l} \delta\theta_1: \\ \delta\theta_2: \\ \delta\theta_3: \end{array} \quad \ell^2 \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \\ \ddot\theta_3 \end{Bmatrix} + \ell^2 \begin{bmatrix} \frac{mg}{\ell}+k & -k & 0 \\ -k & 2\frac{mg}{\ell}+2k & -k \\ & -k & \frac{mg}{\ell}+k \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\longrightarrow \quad [M] = \ell^2 \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}, \qquad [K] = \ell^2 \begin{bmatrix} \frac{mg}{\ell}+k & -k & 0 \\ -k & \frac{2mg}{\ell}+2k & -k \\ 0 & -k & \frac{mg}{\ell}+k \end{bmatrix}$$

For a single pendulum (θ small):



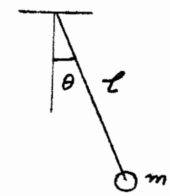
$$\underset{\sim}{\Xi}_0 = \frac{d}{dt}\underset{\sim}{H}_0 \qquad \underset{\sim}{H}_0 = [I]_0 \underset{\sim}{\omega} = m\ell^2\dot\theta\,\hat{\underset{\sim}{e}}_x$$

$$\underset{\sim}{\Xi}_0 = -(mg\ell\theta + k\ell^2\theta)\,\hat{\underset{\sim}{e}}_x$$

Problem 4

$$\Longrightarrow \quad \ddot{\theta} + \left(\frac{g}{\ell} + \frac{k}{m}\right)\theta = 0 \quad \longrightarrow \quad \omega^2 = \frac{g}{\ell} + \frac{k}{m}$$

and if there is no spring : $\omega^2 = \dfrac{g}{\ell}$



So it is easy to guess two of the modes :

$$\{a\}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \omega_1^2 = \frac{g}{\ell}$$

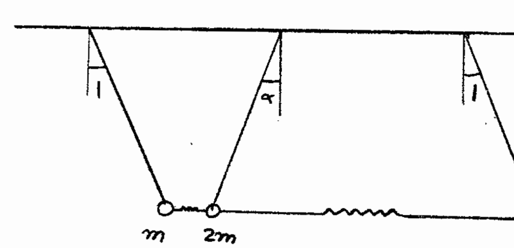
In this mode, springs do not come into play and pendulums oscillate with frequency $\omega = \sqrt{\dfrac{g}{\ell}}$.

$$\{a\}_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \qquad \omega_2^2 = \frac{g}{\ell} + \frac{k}{m}$$

In this mode, mass $2m$ is stationary and outer masses always move in opposite directions.

To find the third mode :

Try $\{a\}_3 = \begin{Bmatrix} 1 \\ -\alpha \\ 1 \end{Bmatrix}$



Using orthogonality of mode shapes :

$$\{a\}_3^t \, [M] \, \{a\}_1 = 0$$

$$\longrightarrow \{1 \quad -\alpha \quad 1\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} (m\ell^2) \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 0 \quad \longrightarrow \quad \alpha = 1$$

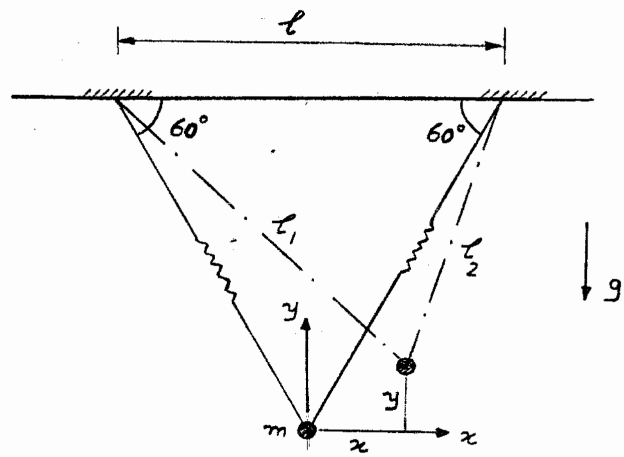
Problem 4

$$\therefore \quad \{a\}_3 = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

$$\omega_3^2 = \frac{\{a\}_3^t \, [K] \, \{a\}_3}{\{a\}_3^t \, [M] \, \{a\}_3} = \frac{\{1 \quad -1 \quad 1\} \begin{bmatrix} \frac{mg}{\ell}+k & -k & 0 \\ -k & 2\frac{mg}{\ell}+2k & -k \\ 0 & -k & \frac{mg}{\ell}+k \end{bmatrix} (\ell^2) \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}}{\{1 \quad -1 \quad 1\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} (m\ell^2) \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}}$$

$$\omega_3^2 = \frac{g}{\ell} + 2\frac{k}{m}$$

Problem 5

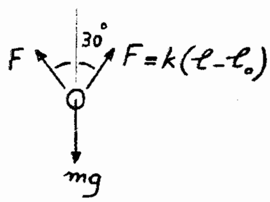


Free length of the springs: ℓ_0

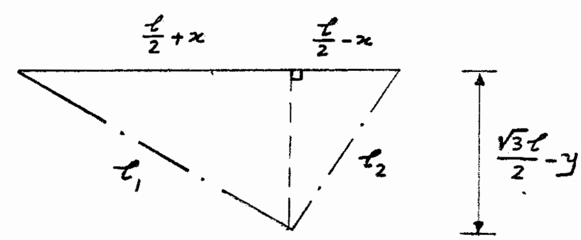
Length of the springs at static equilibrium: ℓ

At static equilibrium:

F.B.D.



$$mg = 2F\cos 30^\circ = \sqrt{3}\,F = \sqrt{3}\,k(\ell - \ell_0)$$



$$\ell_1 = \sqrt{\left(\frac{\ell}{2}+x\right)^2 + \left(\frac{\sqrt{3}}{2}\ell - y\right)^2}$$

$$\ell_2 = \sqrt{\left(\frac{\ell}{2}-x\right)^2 + \left(\frac{\sqrt{3}}{2}\ell - y\right)^2}$$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$V = mgy + \frac{1}{2}k\left(\ell_1 - \ell_0\right)^2 + \frac{1}{2}k\left(\ell_2 - \ell_0\right)^2$$

$$V = mgy + \frac{1}{2}k\left[\ell_1^2 + \ell_2^2 - 2\ell_0\left(\ell_1 + \ell_2\right) + 2\ell_0^2\right]$$

$$V = mgy + k\left[\ell^2 + x^2 + y^2 - \sqrt{3}\,\ell y - \ell_0\left(\ell_1 + \ell_2\right) + \ell_0^2\right]$$

$$\mathcal{L} = T - V \qquad \Xi_x = \Xi_y$$

Problem 5

δx : $$m\ddot{x} + k\left[2x - \ell_0\left(\frac{\partial \ell_1}{\partial x} + \frac{\partial \ell_2}{\partial x}\right)\right] = 0$$

δy : $$m\ddot{y} + k\left[2y - \sqrt{3}\ell - \ell_0\left(\frac{\partial \ell_1}{\partial y} + \frac{\partial \ell_2}{\partial y}\right)\right] + mg = 0$$

$$\frac{\partial \ell_1}{\partial x} = \left(\frac{\ell}{2} + x\right)\left[\left(\frac{\ell}{2}+x\right)^2 + \left(\frac{\sqrt{3}}{2}\ell - y\right)^2\right]^{-\frac{1}{2}}$$

$$\frac{\partial \ell_2}{\partial x} = -\left(\frac{\ell}{2} - x\right)\left[\left(\frac{\ell}{2}-x\right)^2 + \left(\frac{\sqrt{3}}{2}\ell - y\right)^2\right]^{-\frac{1}{2}}$$

$$\frac{\partial \ell_1}{\partial y} = -\left(\frac{\sqrt{3}}{2}\ell - y\right)\left[\left(\frac{\ell}{2}+x\right)^2 + \left(\frac{\sqrt{3}}{2}\ell - y\right)^2\right]^{-\frac{1}{2}}$$

$$\frac{\partial \ell_2}{\partial y} = -\left(\frac{\sqrt{3}}{2}\ell - y\right)\left[\left(\frac{\ell}{2}-x\right)^2 + \left(\frac{\sqrt{3}}{2}\ell - y\right)^2\right]^{-\frac{1}{2}}$$

x, y are small and one can approximate the derivatives using Taylor series expansion:

$$f(a,b) = (a^2+b^2)^{-\frac{1}{2}}$$

$$f(a_0+\Delta a,\ b_0+\Delta b) = \left[(a_0+\Delta a)^2 + (b_0+\Delta b)^2\right]^{-\frac{1}{2}} = f(a_0,b_0) + \frac{\partial f}{\partial a}\Big|_{\substack{a=a_0\\ b=b_0}}\Delta a + \frac{\partial f}{\partial b}\Big|_{\substack{a=a_0\\ b=b_0}}\Delta b$$

$$\approx (a_0^2+b_0^2)^{-\frac{1}{2}} - a_0(a_0^2+b_0^2)^{-\frac{3}{2}}\Delta a - b_0(a_0^2+b_0^2)^{-\frac{3}{2}}\Delta b$$

$\longrightarrow$

$$\frac{\partial \ell_1}{\partial x} = \left(\frac{\ell}{2}+x\right)\left[\ell^{-1} - \frac{\ell^{-2}}{2}x + \frac{\sqrt{3}}{2}\ell^{-2}y + \cdots\right] \approx \frac{1}{2} - \frac{1}{4}\frac{x}{\ell} + \frac{\sqrt{3}}{4}\frac{y}{\ell} + \frac{x}{\ell}$$

$$\left(a_0 = \frac{\ell}{2},\quad b_0 = \frac{\sqrt{3}}{2}\ell,\quad \Delta a = x,\quad \Delta b = -y\right)$$

$$\frac{\partial \ell_2}{\partial x} \approx -\left(\frac{1}{2} + \frac{1}{4}\frac{x}{\ell} + \frac{\sqrt{3}}{4}\frac{y}{\ell} - \frac{x}{\ell}\right) \qquad \left(a_0 = \frac{\ell}{2},\quad b_0 = \frac{\sqrt{3}}{2}\ell,\quad \Delta a = -x,\quad \Delta b = -y\right)$$

$$\frac{\partial \ell_1}{\partial y} = -\left(\frac{\sqrt{3}}{2}\ell - y\right)\left[\ell^{-1} - \frac{\ell^{-2}}{2}x + \frac{\sqrt{3}}{2}\ell^{-2}y + \cdots\right] \approx -\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\frac{x}{\ell} - \frac{3}{4}\frac{y}{\ell} + \frac{y}{\ell}$$

Problem 5

$$\frac{\partial l_2}{\partial y} \approx -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\frac{x}{\ell} - \frac{3}{4}\frac{y}{\ell} + \frac{y}{\ell}$$

$\therefore$

$$\begin{array}{l} \underline{\underline{\delta x}}: \\ \\ \underline{\underline{\delta y}}: \end{array} \quad \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + k \begin{bmatrix} 2 - \frac{3}{2}\frac{\ell_0}{\ell} & 0 \\ 0 & 2 - \frac{1}{2}\frac{\ell_0}{\ell} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(Note that $mg - \sqrt{3}k(\ell - \ell_0) = 0$ in δy)

$$\Longrightarrow \quad [M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \qquad [K] = \begin{bmatrix} 2 - \frac{3}{2}\frac{\ell_0}{\ell} & 0 \\ 0 & 2 - \frac{1}{2}\frac{\ell_0}{\ell} \end{bmatrix} k$$

$$-\omega^2[M] + [K] = 0 \quad \Longrightarrow \quad \begin{cases} \omega_1^2 = \left(2 - \frac{3}{2}\frac{\ell_0}{\ell}\right)\frac{k}{m}, & \{a\}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \\ \omega_2^2 = \left(2 - \frac{1}{2}\frac{\ell_0}{\ell}\right)\frac{k}{m}, & \{a\}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \end{cases}$$