I-campus project
School-wide Program on Fluid Mechanics
Modules on Waves in fluids

T. R. Akylas & C. C. Mei

CHAPTER SEVEN
INTERNAL WAVES IN A STRATIFIED FLUID

1 Introduction.

The atmosphere and ocean are continuously stratified due to change in temperature, composition and pressure. These changes in the ocean and atmosphere can lead to significant variations of density of the fluid in the vertical direction. As an example, fresh water from rivers can rest on top of sea water, and due to the small diffusivity, the density contrast remains for a long time. The density stratification allows oscillation of the fluid to happen. The restoring force that produces the oscillation is the buoyancy force. The wave phenomena associated with these oscillations are called internal waves and are discussed in this chapter.

2 Governing Equations for Incompressible Density-stratified Fluid.

We are going to derive the system of equations governing wave motion in an incompressible fluid with continuous density stratification. Cartesian coordinates x, y and z will be used, with z measured vertically upward. The velocity components in the directions of increasing x, y and z will be denoted as u, v and w. The fluid particle has to satisfy the continuity equation

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.1}$$

and the momentum equations

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \quad -\frac{\partial p}{\partial x}, \tag{2.2}$$

$$\rho\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \quad -\frac{\partial p}{\partial y}, \tag{2.3}$$

$$\rho\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \quad -\frac{\partial p}{\partial z} - g\rho, \tag{2.4}$$

where ρ and p are, respectively, the fluid density and pressure. The fluid is taken to be such that the density depends only on entropy and on composition, i.e., ρ depends only on the potential temperature θ and on the concentrations of constituents, e.g., the salinity s or humidity q. Then for fixed θ and q (or s), ρ is *independent of pressure*:

$$\rho \quad \rho(\theta, q). \tag{2.5}$$

The motion that takes place is assumed to be isentropic and without change of phase, so that θ and q are constant for a material element. Therefore

$$\frac{D\rho}{Dt} \quad \frac{\partial \rho}{\partial \theta}\frac{D\theta}{Dt} + \frac{\partial \rho}{\partial q}\frac{Dq}{Dt} \quad . \tag{2.6}$$

In other words, ρ is constant for a material element because θ and q are, and ρ depends only on θ and q. Such a fluid is said to be *incompressible*, and because of (2.6) the continuity equation (2.1) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad . \tag{2.7}$$

For an incompressible fluid, the density ρ satisfies the density equation

$$\frac{1}{\rho}\frac{D\rho}{Dt} \quad . \tag{2.8}$$

Assuming that the velocities are small, we can linearize the momentum equations to obtain

$$\rho\frac{\partial u}{\partial t} \quad -\frac{\partial p}{\partial x}, \tag{2.9}$$

$$\rho\frac{\partial v}{\partial t} \quad -\frac{\partial p}{\partial y}, \tag{2.10}$$

$$\rho\frac{\partial w}{\partial t} \quad -\frac{\partial p}{\partial z} - g\rho. \tag{2.11}$$

Next, we consider that the wave motion results from the perturbation of a state of equilibrium, which is the state of rest. So the distribution of density and pressure is the hydrostatic equilibrium distribution given by

$$\frac{\partial \bar{p}}{\partial z} \quad -g\bar{\rho}. \tag{2.12}$$

When the motion develops, the pressure and density changes to

$$p \quad p(z) + p', \qquad\qquad =^{-} \tag{2.13}$$

$$\rho \quad \rho(z) + \rho', \qquad\qquad =^{-} \tag{2.14}$$

where p' and ρ' are, respectively, the pressure and density perturbations of the "background" state in which the density $\bar{\rho}$ and the pressure $\bar{p}$ are in hydrostatic balance. The density equation now assumes the form

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{\partial \bar{\rho}}{\partial z} + w\frac{\partial \rho'}{\partial z} \qquad . \tag{2.15}$$

The nonlinear terms $u(\partial\rho'/\partial x)$, $v(\partial\rho'/\partial y)$ and $w(\partial\rho'/\partial z)$ are negligible for small amplitude motion, so the equation (2.15) simplifies to

$$\frac{\partial \rho'}{\partial t} + w\frac{\partial \bar{\rho}}{\partial z} \qquad , \tag{2.16}$$

which states that the density perturbation at a point is generated by a vertical advection of the background density distribution. The continuity equation (2.7) for incompressible fluid stays the same, but the momentum equations (2.9) to (2.11) assume the form

$$\bar{\rho}\frac{\partial u}{\partial t} \qquad -\frac{\partial p'}{\partial x}, \tag{2.17}$$

$$\bar{\rho}\frac{\partial v}{\partial t} \qquad -\frac{\partial p'}{\partial y}, \tag{2.18}$$

$$\bar{\rho}\frac{\partial w}{\partial t} \qquad -\frac{\partial p'}{\partial z} - g\rho'. \tag{2.19}$$

We would like to reduce the systems of equations (2.7), (2.16) and (2.17) to (2.19) to a single partial differential equation. This can be achieved as follows. First, we take the time derivative of the continuity equation to obtain

$$\frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 v}{\partial t \partial y} + \frac{\partial^2 w}{\partial t \partial z} \qquad . \tag{2.20}$$

Second, we take the x, y and t derivatives, respectively, of the equations (2.17) to (2.19), and we obtain

$$\bar{\rho}\frac{\partial^2 u}{\partial x \partial t} \qquad -\frac{\partial^2 p'}{\partial x^2}, \tag{2.21}$$

$$\bar{\rho}\frac{\partial^2 v}{\partial y \partial t} \qquad -\frac{\partial^2 p'}{\partial y^2}, \tag{2.22}$$

$$\bar{\rho}\frac{\partial^2 w}{\partial t^2} \qquad -\frac{\partial^2 p'}{\partial t \partial z} - g\frac{\partial \rho'}{\partial t}. \tag{2.23}$$

If we substitute equations (2.21) and (2.22) into equation (2.20), we obtain

$$-\frac{1}{\bar{\rho}}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right) + \frac{\partial^2 w}{\partial t \partial z} \qquad . \tag{2.24}$$

We can eliminate ρ' from (2.23) by using equation (2.16) to obtain

$$\bar{\rho}\frac{\partial^2 w}{\partial t^2} \qquad -\frac{\partial^2 p'}{\partial t \partial z} + g\frac{\partial \bar{\rho}}{\partial z}w. \tag{2.25}$$

Third, we apply the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ to equation (2.25) to obtain

$$\bar{\rho}\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \qquad -\frac{\partial^2}{\partial t \partial z}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2}\right) + g\frac{\partial \bar{\rho}}{\partial z}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \tag{2.26}$$

Next, we use equation (2.24) to eliminate p' from equation (2.26), which gives the following partial differential equation for w:

$$\frac{\partial^2}{\partial t^2} \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left[\bar{\rho}\frac{\partial w}{\partial z}\right] \quad + N^2 \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \qquad , \tag{2.27}$$

where we define

$$N^2(z) = \ -\frac{g}{\rho}\frac{\partial \bar{\rho}}{\partial z}, \tag{2.28}$$

which has the units of frequency (rad/sec) and is called the Brunt-Väisälä frequency or buoyancy frequency. If we assume that w varies with z much more rapidly than $\bar{\rho}(z)$, then

$$\frac{1}{\bar{\rho}}\frac{\partial}{\partial z} \quad \bar{\rho}\frac{\partial}{\partial z} \quad w \sim \frac{\partial^2 w}{\partial z^2}, \tag{2.29}$$

and (2.27) can be approximated by the equation

$$\frac{\partial^2}{\partial t^2} \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \quad + N^2 \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \qquad . \tag{2.30}$$

The assumption above is equivalent to the Boussinesq approximation, which applies when the motion has vertical scale small compared with the scale of the background density. It consists in taking the density to be constant in computing rates of change of momentum from accelerations, but taking full account of the density variations when they give rise to buoyancy forces, i.e., when there is a multiplying factor g in the vertical component of the momentum equations. The Boussinesq approximation leads to equation (2.30) for the vertical velocity w.

3 The Buoyancy Frequency (Brunt-Väisälä frequency).

Consider a calm stratified fluid with a static density distribution $\bar{\rho}(z)$ which decreases with height z. If a fluid parcel is moved from the level z upward to $z+\zeta$, it is surrounded by lighter fluid of density $\bar{\rho}(z+\zeta)$. The upward buoyancy force per unit volume is

$$g\left[\bar{\rho}(z+\zeta)-\bar{\rho}(z)\right] \approx g\frac{d\bar{\rho}}{dz}\zeta, \tag{3.31}$$

and it is negative. Applying Newton's law to the fluid parcel of unit volume, we have

$$\bar{\rho}\frac{\partial^2\zeta}{\partial t^2} \quad g\frac{d\bar{\rho}}{dz}\zeta \tag{3.32}$$

or

$$\frac{\partial^2\zeta}{\partial t^2}+N^2\zeta \quad\quad , \tag{3.33}$$

where

$$N^2(z)= \; -\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz}, \tag{3.34}$$

which is called the buoyancy frequency or the Brunt Väsälä frequency. This elementary consideration shows that once a fluid is displaced from its equilibrium position, gravity and density gradient provide restoring force to enable oscillations.

4 Internal Gravity Waves in Unbounded Stratified Fluid.

Consider the case in which the buoyancy (Brunt-Väisälä) frequency N is constant throughout the fluid. Traveling wave solutions of (2.30) can be found of the form

$$w \quad w_0\cos(kx+ly+mz-\omega t), \tag{4.35}$$

where w_0 is the vertical velocity amplitude and $\overrightarrow{k} \quad k,l,m)$ is the wavenumber of the disturbance, and ω is the frequency. In order for (4.35) to satisfy the governing equation (2.30) for the vertical perturbation velocity, ω and $\overrightarrow{k}$ must be related by the dispersion relation

$$\omega^2 \quad \frac{(k^2 + l^2)N^2}{k^2 + l^2 + m^2}. \tag{4.36}$$

Thus internal waves can have any frequency between zero and a maximum value of N. The dispersion relation for internal waves is of quite a different character compared to that for surface waves. In particular, the frequency of surface waves depends only on the magnitude $|\overrightarrow{k}|$ of the wavenumber, whereas the frequency of internal waves is independent of the magnitude of the wavenumber and depends only on the angle ϕ that the wavenumber vector makes with the horizontal. To illustrate this, we consider the spherical system of coordinates in the wavenumber space, namely,

$$k \quad |\overrightarrow{k}| \cos(\phi) \cos(\theta) \tag{4.37}$$

$$l \quad |\overrightarrow{k}| \cos(\phi) \sin(\theta) \tag{4.38}$$

$$m \quad |\overrightarrow{k}| \sin(\phi) \tag{4.39}$$

The coordinate system in the wavenumber space is given in the figure 1.

The dispersion relation given by equation (4.36) reduces to

$$\omega^2 \quad N \cos(\phi). \tag{4.40}$$

Now we can write expressions for the quantities p', ρ', u and v. From equation (2.20) we can write

$$-\frac{1}{\rho_0} \quad \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} \qquad \frac{\partial^2 w}{\partial t \partial z} \quad \omega m w_0 \cos(kx + ly + mz - \omega t),$$

which implies that the perturbation pressure p' is given by

$$p' \quad -\frac{\omega m w_0 \rho_0}{(k^2 + l^2)^{1/2}} \cos(kx + ly + mz - \omega t). \tag{4.41}$$

From equation (2.16) we have the perturbation density ρ' given by

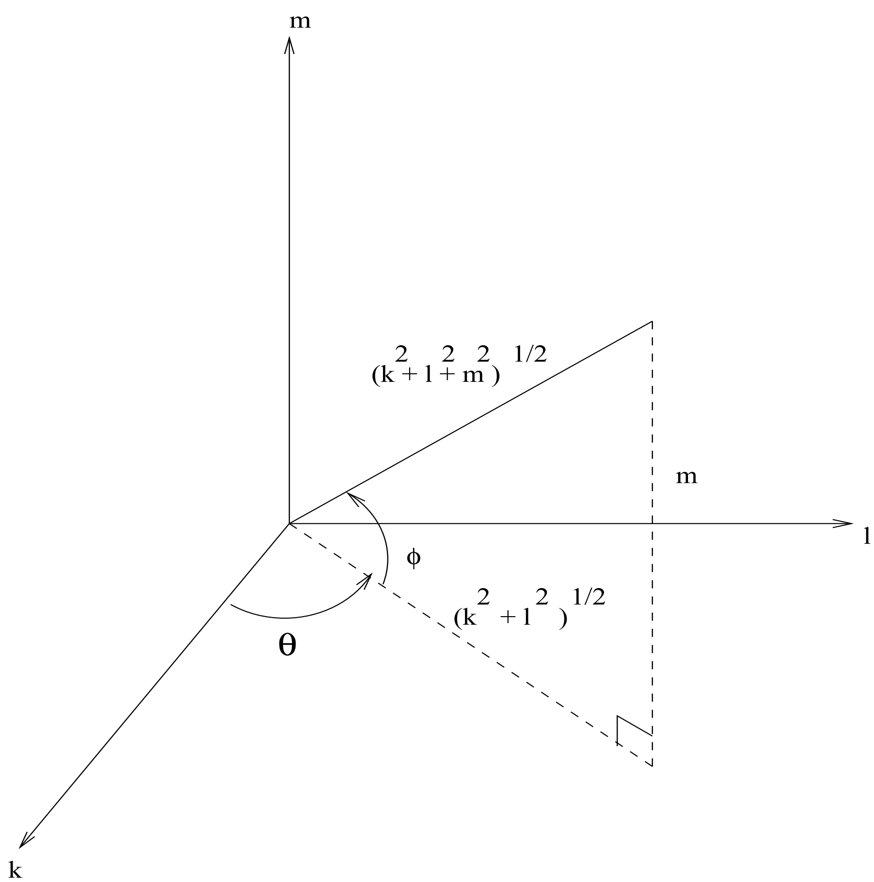

Figure 1: Coordinate system in the wavenumber space.

$$\rho' \quad - \quad \frac{N^2}{\omega g} \quad \rho_0 w_0 \sin(kx + ly + mz - \omega t). \qquad (4.42)$$

The horizontal velocity components can be found from equations (2.17) and (2.18), which give

$$(u, v) = \; -(k, l)(k^2 + l^2)^{-1} m w_0 \cos(kx + ly + mz - \omega t) \qquad (4.43)$$

$$k, l)(\omega \rho_0)^{-1} p'. \qquad (4.44) \quad =($$

The above relations between pressure and velocity fluctuations can be useful for deducing wave properties from observations at a fixed point. For instance, if the horizontal velocity components and perturbation pressure of a progressive wave are measured, the horizontal component of the wavenumber vector can be deduced from (4.44).

A sketch showing the properties of a plane progressive internal wave in the vertical plane that contains the wavenumber vector is presented in figure 2. The particle motion is along wave crests, and there is no pressure gradient in this direction. The restoring force on a particle is therefore due solely to the component $g \cos \phi$ of gravity in the direction of motion. The restoring force is also proportional to the component of density change in this direction, which is $\cos \phi \frac{d\rho}{dz}$ per unit displacement.

Consider now the succession of solutions as ϕ progressively increases from zero to $\pi/2$. When ϕ , a vertical line of particles moves together like a rigid rod undergoing =0 longitudinal vibrations. When the line of particles is displaced from its equilibrium, buoyancy restoring forces come into play just as if the line of particles were on a spring, resulting in oscillations of frequency N. The solution for increasing values of ϕ correspond to lines of particles moving together at angle ϕ to the vertical. The restoring force per unit displacement ($\cos \phi d\rho'/dz$) is less than the case where ϕ , so the frequency of vibration is less. As ϕ tends to $\pi/2$, the frequency of vibration tends to zero. The case $\phi \quad \pi/2$ is not an internal wave, but it represents an important form of motion that is often observed. For instance, it is quite common on airplane journeys to see thick layers of cloud that are remarkably flat and extensive. Each cloud layer is moving in its own horizontal plane, but different layers are moving relative to each other.

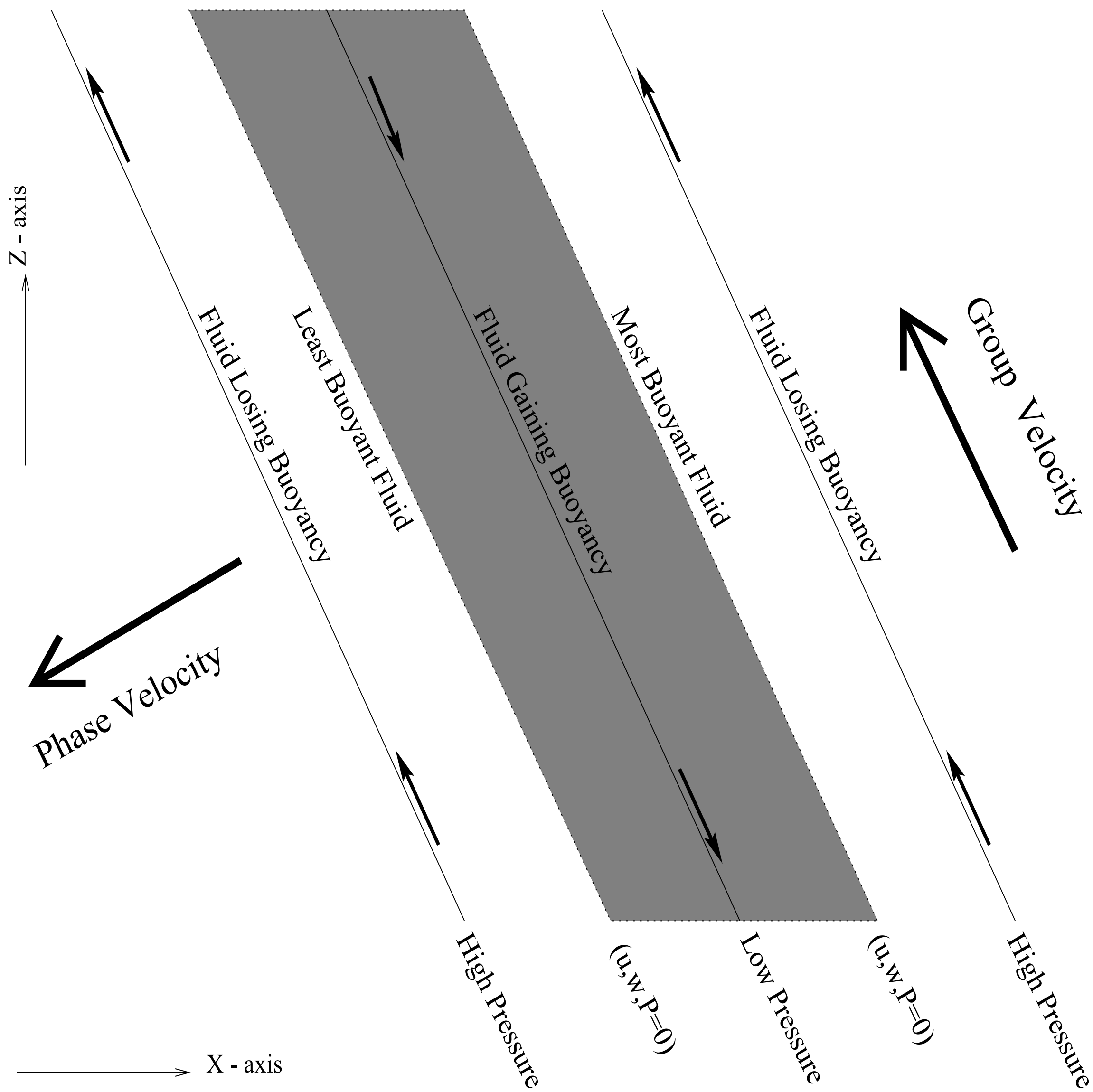

Figure 2: The instantaneous distribution of velocity, pressure, and buoyancy perturbations in an internal gravity wave. This is a view in the x, z plane. The phase of the wave is constant along the slanting, dashed, and solid lines. Velocity and pressure perturbations have extrema along the solid lines; buoyancy perturbations are zero along the solid lines. Buoyancy perturbations have extrema, and velocity and pressure perturbations are zero along dashed lines. Small arrows indicate the perturbation velocities, which are always parallel to the lines of constant phase. Large heavy arrows indicate the direction of phase propagation and group velocity.

4.1 Dispersion Effects.

In practice, internal gravity waves never have the form of the exact plane wave given by equation (4.35), so it is necessary to consider superposition of such waves. As a consequence, dispersion effects become evident, since waves with different frequencies have different phase and group velocities as we are going to show in this section. For internal waves, surfaces of constant frequency in the wavenumber space are the cones $\phi \quad$ constant. The phase velocity is parallel to the wavenumber vector and it lies on a cone of constant phase. Its magnitude is

$$\frac{\omega}{|\overrightarrow{k}|} \quad \left(\frac{N}{|\overrightarrow{k}|}\right) \cos\phi. \tag{4.45}$$

The *group velocity* C_g is the gradient of the frequency ω in the wavenumber space and therefore is normal to the surface of constant frequency ω. It follows that the group velocity is at right angles to the wavenumber vector. When the group velocity has an upward component, therefore, the phase velocity has a downward component, and vice versa. The group velocity vector is

$$C_g \quad \frac{N}{|\overrightarrow{k}|} \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, -\cos\phi). \tag{4.46}$$

Therefore, the magnitude of the group velocity is $(\frac{N}{|\overrightarrow{k}|})\sin\phi$, and its direction is at an angle ϕ to the vertical, as illustrated in the figure 3.

To illustrate the effects of dispersion, we consider the case of two dimensional motions. We consider only the coordinates x and z. In this case, the wavenumber is the vector (k, m). We consider an initially localized wave packet. Due to dispersion effects, the wave packet spreads and moves with the group velocity vector C_g, which now simplifies to

$$C_g \quad \frac{N}{|\overrightarrow{k}|} \sin\phi(\sin\phi, -\cos\phi). \tag{4.47}$$

The phase velocity is perpendicular to the group velocity vector, so the wave crests (lines of constant phase) move perpendicularly to the direction of propagation of the

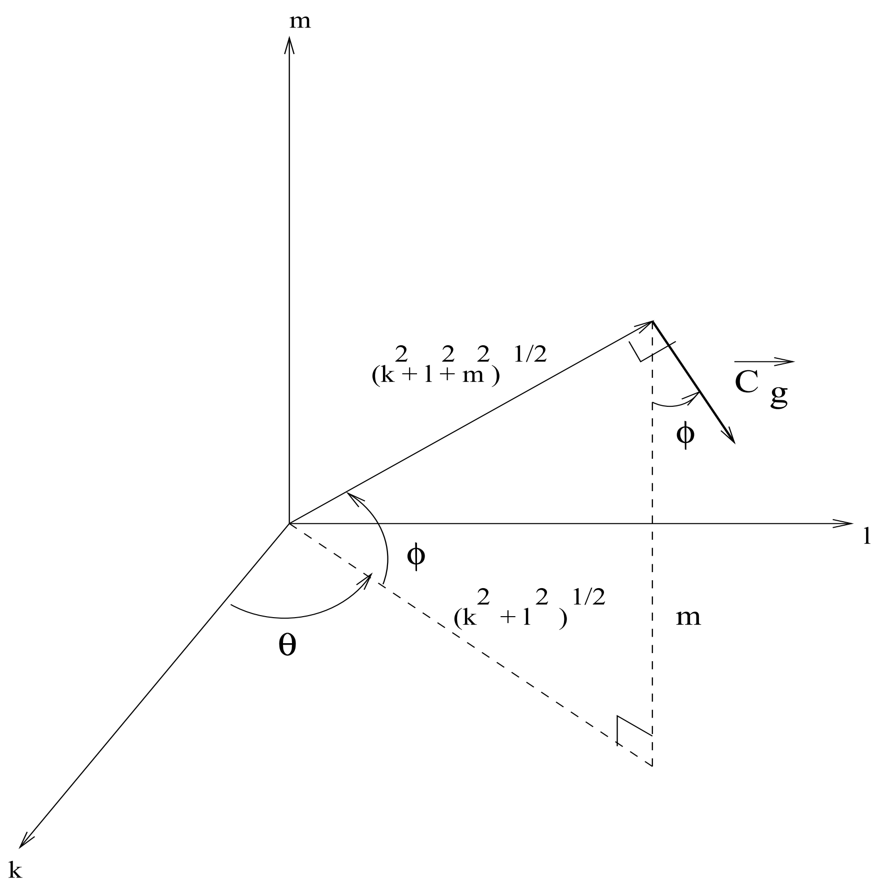

Figure 3: Wavenumber vector and group velocity vector.

wave packet. The phase velocity is given by the equation (4.45), where the wavenumber vector $\overrightarrow{k}$ makes an angle ϕ with the horizontal direction (see figure 1, but now set $\theta \quad 0$).

To illustrate the effects of dispersion, we consider three different animations of a localized wave packet for the density perturbation ρ'. The perturbation density ρ' is related to the vertical velocity w by the equation

$$\frac{\partial \rho'}{\partial t} \qquad \frac{\rho_0 N^2}{g} w, \tag{4.48}$$

and the governing equation for the vertical velocity w is given by the equation (2.30). To obtain the evolution in time of an initially localized wave packet for the perturbation density, we apply a two-dimensional Fourier transform to equations (2.30) and (4.48). The two-dimensional Fourier transform pair considered is

$$\hat{u}(k,m) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dz \left\{ \exp(-ikx - imz) u(x,z) \right\} \tag{4.49}$$

and

$$u(x,z) = \frac{1}{4 \quad ^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dm \left\{ \exp(-ikx - imz) \hat{u}(k,m) \right\}. \tag{4.50}$$

The Fourier transform of the equation (2.30) is given by the equation

$$\frac{\partial^2 \hat{w}}{\partial t^2} + \frac{N^2 k^2}{k^2 + m^2} \hat{w} \qquad , \tag{4.51}$$

which has solution of the form

$$\hat{w}(k,m,t) = A(k,m) \exp(i\omega t) + B(k,m) \exp(-i\omega t), \tag{4.52}$$

where ω is given by the dispersion relation

$$\omega \qquad \frac{Nk}{\sqrt{k^2 + m^2}}. \tag{4.53}$$

The Fourier transform of the equation (4.48) is given by the equation

$$\frac{\partial \hat{\rho}'}{\partial t} \quad \frac{\rho_0 N^2}{g}\hat{w}. \tag{4.54}$$

From equations (4.51) and (4.54) we have that

$$\hat{\rho}'(k,m,t)= \frac{\rho_0 N^2}{g\omega(k,m)}\left\{-iA(k,m)\exp(i\omega t)+iB(k,m)\exp(i\omega t)\right\}, \tag{4.55}$$

where the constants A and B are determined from the Fourier transform of the initial conditions for ρ', given by the equations

$$\rho'(x,z,0) \quad f(x,z), \tag{4.56}$$

$$\frac{\partial \rho'}{\partial t}(x,z,0) =0, \tag{4.57}$$

which implies that the constants $A(k,m)$ and $B(k,m)$ are given by the equations

$$A(k,m)= \frac{ig\omega}{2\rho_0 N^2}\hat{f}, \tag{4.58}$$

$$A(k,m)= -\frac{ig\omega}{2\rho_0 N^2}\hat{f}. \tag{4.59}$$

The perturbation density $\rho'(x,z,t)$ is finally given by the equation

$$\rho'(x,z,t)= \frac{1}{8}\int_{-\infty}^{\infty}\frac{dk}{\pi g}\int_{-\infty}^{\infty} dm \left\{\hat{f}(k,m)\exp(i\omega(k,m)t)+\hat{f}(k,m)\exp(-i\omega(k,m)t)\right\} \exp(-ikx-imz). \tag{4.60}$$

The function $f(x,z)$ and its Fourier transform $\hat{f}(k,m)$ are given by the equations

$$f(x,z)= \frac{1}{2}\exp \quad -\frac{1}{2}x^2\sigma^2-\frac{1}{2}z^2\tau^2 \quad \cos(\tilde{k}x+\tilde{m}z), \tag{4.61}$$

$$\hat{f}(k,m)= \frac{1}{2\sigma\tau}\left\{\exp \quad -\frac{1}{2}\frac{(k-\tilde{k})^2}{\sigma^2}-\frac{1}{2}\frac{(m-\tilde{m})^2}{\tau^2}\Bigg) + \exp \quad -\frac{1}{2}\frac{(k+\tilde{k})^2}{\sigma^2}-\frac{1}{2}\frac{(m+\tilde{m})^2}{\tau^2}\Bigg)\right\}. \tag{4.62}$$

In the animation which follows, we show the results from the numerical evaluation of the inverse Fourier transform in equation (4.60) for a sequence of values of the variable t with $\hat{f}$ given by equation (4.62).

The first animation has as initial condition a Gaussian wave packet with σ 1/4, τ /4 and $\tilde{k}$ $\tilde{m}$ $\frac{\pi}{2}$. This initial wave packet has a circular shape and splits in two parts as time increases. This two parts propagate in opposite directions from each other. Since the x and z components of the main wavenumber are equal and positive and the wave packet has the same modulation along the x and z directions (σ τ), the two parts of the initial wave packet travel towards the middle of the second and fourth quadrants, as we see in the animation. For the wave packet in the second (fourth) quadrant the group velocity vector points away from the origin towards the middle of the second (fourth) quadrant, so the phase velocity, which is orthogonal to the group velocity, is oriented in the anti-clockwise (clockwise) sense, as we can see from the crests movement in the animation. When the two parts resulting from the initial wave packet are still close, we see some constructive and destructive interference. To see this animation, click here.

The second animation has as initial condition a Gaussian wave packet with σ /2, τ /100 and $\tilde{k}$ $\tilde{m}$ $\frac{\pi}{2}$. This initial wave packet has a shape of an elongated ellipse in the x direction. In the movie frame, this initial wave packet looks almost without variation in the x direction. The wave packet splits in two parts as time increases. These two parts propagate in opposite directions from each other, in a way similar to the previous example. The interference effect between these two wave packets for early times is more intense than what was observed in the previous example, as we can see in the animation. To see it, click here.

The third animation has as initial condition a Gaussian wave packet with σ /2, τ /20 and $\tilde{k}$ $\tilde{m}$ $\frac{\pi}{2}$. This initial wave packet has a shape of an elongated ellipse in the x direction. The movie frame shows the whole wave packet, which splits in two parts as time increases. This two parts propagate in opposite directions from each other, but the group velocity vector has a slightly smaller component in the x direction. This is due to the difference of the modulation of the wave packet in the x and z directions, as we can see in the animation. To see this animation, click here.

4.2 Saint Andrew's Cross.

Here we discuss the wave pattern for internal waves produced by a localized source on a sinusoidal oscillation, like an oscillating cylinder for example, in a fluid with constant density gradient (the buoyancy frequency is constant). For sinusoidal internal waves, the wave energy flux $\overrightarrow{I} \quad p'\overrightarrow{u}$ (the perturbation pressure p' is given by equation (4.41) and the components of the velocity vector are given by equations (4.44) and (4.35)) averaged over a period is given by the equation

$$\overrightarrow{I} \qquad \frac{1}{2}\frac{w_0^2 N m \rho_0}{k^2+l^2}\left\{\sin\phi\cos\theta, \sin\phi\sin\theta, -\cos\phi\right\}, \tag{4.63}$$

which is parallel to the group velocity, according to equation (4.46). Therefore, for internal waves the energy propagates in the direction of the group velocity, which is parallel to the surfaces of constant phase. This fact means that internal waves generated by a localized source could never have the familiar appearance of concentric circular crests centered on the source, as we see, for example, for gravity surface waves. Instead, the crests and other surfaces of constant phase stretch radially outward from the source because wave energy travels with the group velocity, which is parallel to surfaces of constant phase.

For a source of definite frequency $\omega \leq N$ (less than the buoyancy frequency), those surfaces are all at a definite angle

$$\phi \qquad \mathrm{os}^{-1}(\omega/N), \tag{4.64}$$

to the *vertical*; therefore, all the wave energy generated in the source region travels at that angle to the *vertical*. Accordingly, it is confined to a double cone with semi-angle ϕ. The direction of the group velocity vector along the double cone is specified by the fact that energy has to radiate out from the source. The direction of propagation of the lines of constant phase is also specified in terms of the direction of the group velocity and by the fact that the phase velocity

$$\overrightarrow{C} \qquad \frac{N\cos\phi}{|\overrightarrow{k}|}\left\{\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi\right\} \tag{4.65}$$

is orthogonal to the group velocity, and that

$$\vec{C} + \vec{C}_g \qquad \frac{N}{|\vec{k}|} \{\cos\theta, \sin\theta, 0\} . \tag{4.66}$$

Then, given the direction of the group velocity, the orthogonality of the phase and group velocity plus the condition (4.66), the direction of the phase velocity is specified. If the group velocity has a positive vertical component, the phase velocity has a negative vertical component and vice-versa. The two-dimensional case of an oscillating cylinder is illustrated in figure 4.

This unique property of anisotropy has been verified in dramatic experiments by Mowbray and Stevenson. By oscillating a long cylinder at various frequencies vertically in a stratified fluid, equal phase lines are only found along four beams forming "St. Andrew's Cross", see figure 5 for ω/N .7 and ω/N .9. It can be verified that the angles are ϕ 45 degrees for ω/N 0.7, and ϕ 26 degrees for ω/N 0.9, in close accordance with the condition (4.64).

5 Waveguide behavior.

In this section we study free wave propagation in a continuously stratified fluid in the presence of boundaries. Attention is restricted to the case in which the bottom is flat. The equilibrium state that is being perturbed is the one at rest, so density, and hence buoyancy frequency, is a function only of the vertical coordinate z. We start with an ocean, which has an upper boundary. The atmosphere is somewhat different from the ocean since it has no definite upper boundary, so a study of waves in this situation is made later in this section.

5.1 The oceanic waveguide

Since we assume the undisturbed state as the state of rest, fluid properties are constant on horizontal surfaces and, furthermore, the boundaries are horizontal. Solutions of the perturbation equation (2.27) can be found in the form

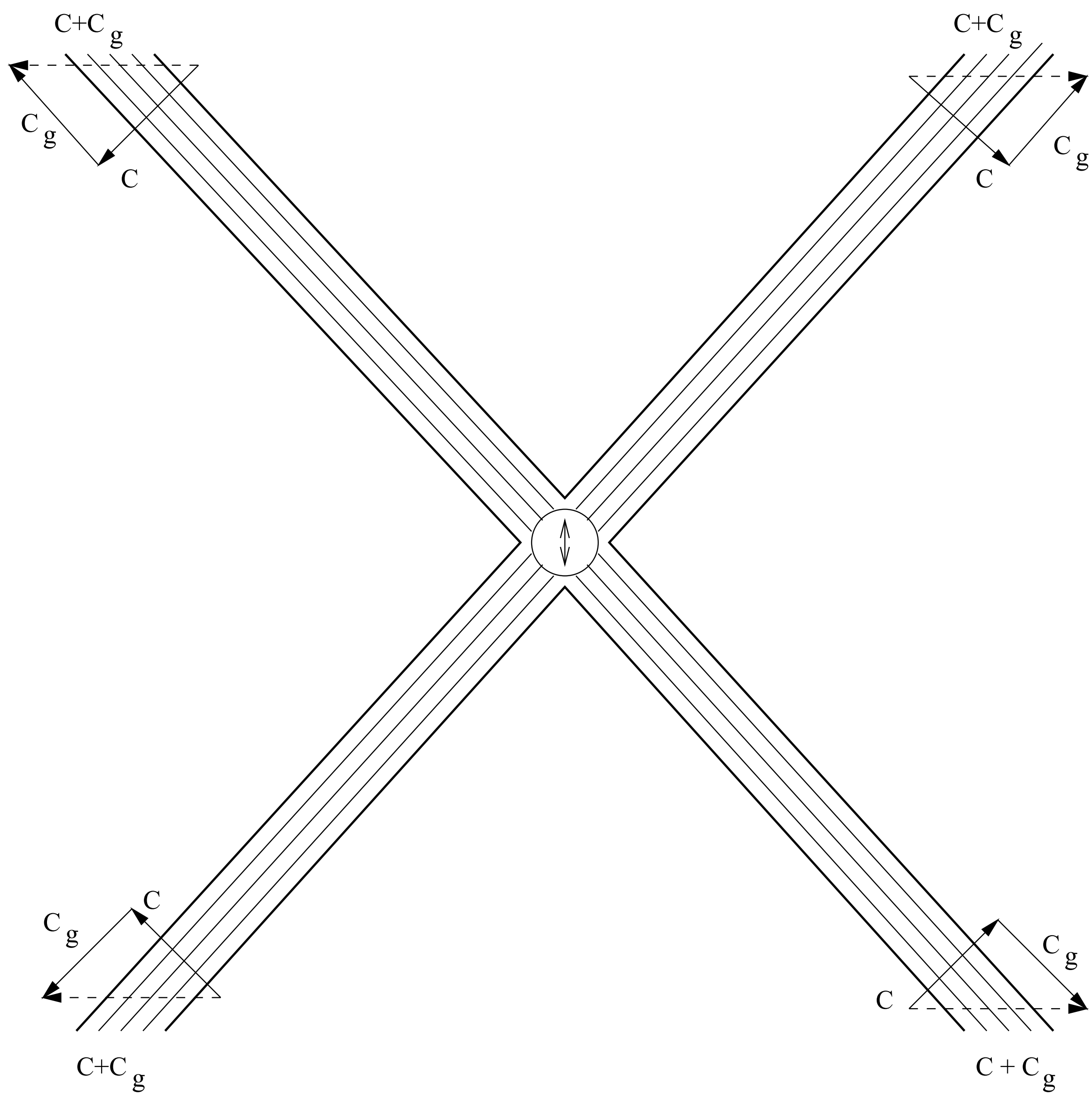

Figure 4: Phase and group velocities.

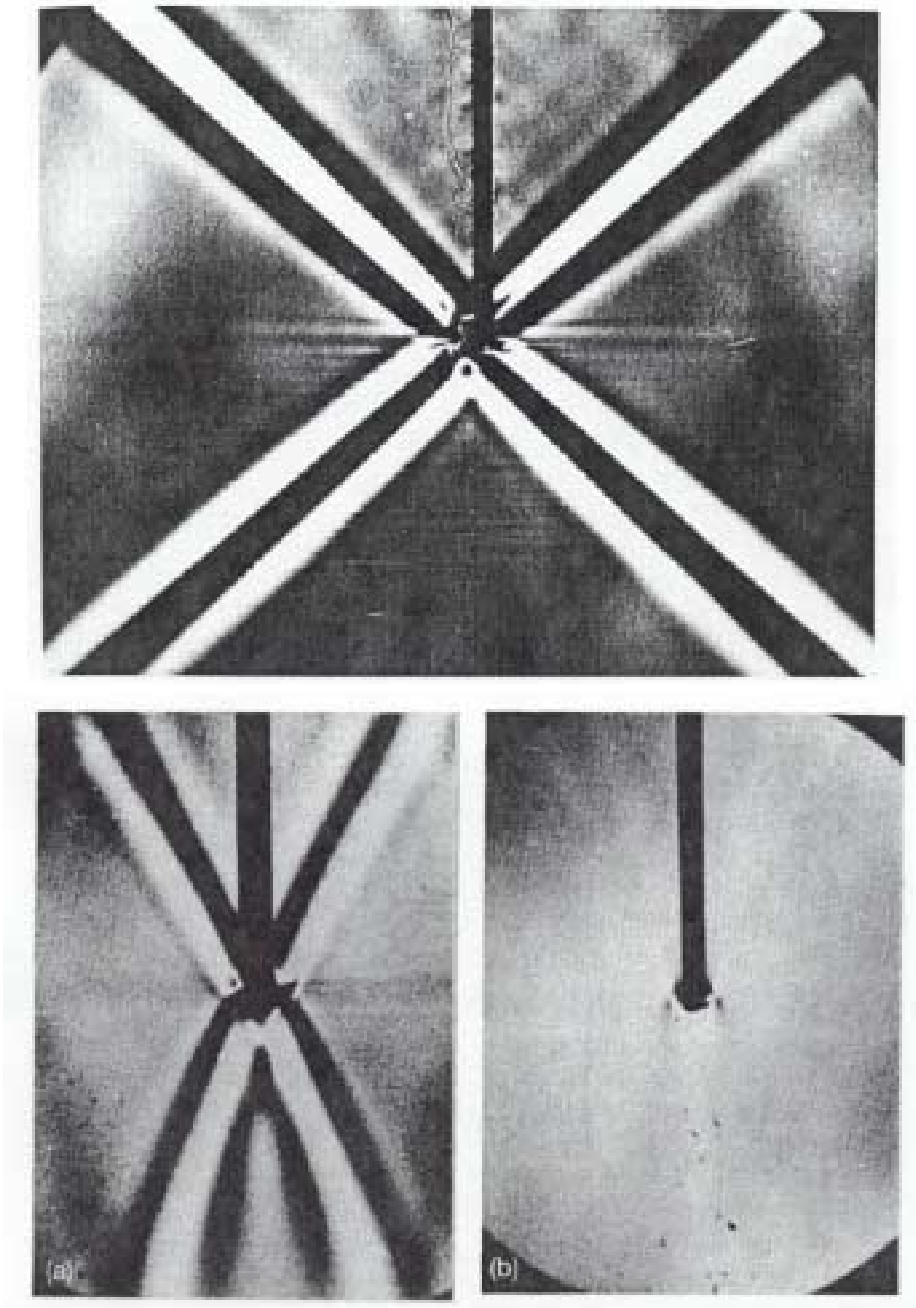

Figure 5: St Andrew's Cross in a stratified fluid. In the top figure ω/N 0.9 and in the left bottom figure ω/N .7. =0

$$w(x,y,z,t) = \ \hat{w}(z)\exp[i(kx+ly-\omega t)] \tag{5.67}$$

The equation for $\hat{w}(z)$ can be found by substitution of equation (5.67) into the governing equation (2.27). We obtain

$$\frac{1}{\bar{\rho}}\frac{\partial}{\partial z} \quad \bar{\rho}\frac{\partial \hat{w}}{\partial z} \quad + \frac{(N^2-\omega^2)}{\omega^2}(k^2+l^2)\hat{w}(z) \quad 0 \tag{5.68}$$

The boundary conditions for this equation are the bottom condition of no flux across it, given by the equation

$$\hat{w}(z) = \ 0 \text{ at } z \quad -H, \tag{5.69}$$

and at the free-surface we have the linearized condition

$$\frac{\partial p'}{\partial t} \quad \bar{\rho} g w(z) \text{ at } z \quad , \tag{5.70}$$

where p' is the perturbation pressure. From this equation we can obtain a free-surface boundary condition for $\hat{w}(z)$. We apply the operator $\frac{\partial}{\partial t}$ to the equation (2.24), and then we substitute equation (5.70) into the resulting equation. As a result, we obtain the equation

$$\frac{\partial^3 w}{\partial t^2 \partial z} \quad g \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \quad \text{at } z \tag{5.71}$$

Now, if we substitute equation (5.67) into the equation (5.71), we obtain the free-surface boundary condition for $\hat{w}(z)$, which follows

$$\frac{\partial \hat{w}}{\partial z} + \frac{g}{\omega^2}(k^2+l^2)\hat{w}(z) = \ 0 \text{ at } z \quad . \tag{5.72}$$

To simplify the governing equation for $\hat{w}(z)$, we make the Boussinesq approximation, such that equation (5.68) simplifies to

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \frac{(N^2 - \omega^2)}{\omega^2}(k^2 + l^2)\hat{w}(z) = 0, \tag{5.73}$$

with boundary conditions given by equations (5.72) and (5.69). The two boundary (bottom and free-surface) have the effect of confining the wave energy to a region of finite extent, so the ocean can be considered as a *waveguide* that causes the energy to propagate horizontally.

A useful piece of imaginary is to picture internal waves propagating obliquely through the ocean, reflections at the upper and lower boundaries ensuing no loss of energy from the wave guide, whereas horizontal propagation is uninhibited.

Next, we obtain the general solution of equation (5.73) under the boundary conditions (5.72) and (5.69). We first consider the case where $\omega^2 > N^2$. For this case the general solution has the form

$$\hat{w}(z) = \frac{\sinh[m(z+H)]}{\sinh(mH)} \text{ with } m^2 \quad \frac{(\omega^2 - N^2)}{\omega^2}(k^2 + l^2), \tag{5.74}$$

which already satisfies the bottom boundary condition. The free-surface boundary condition (5.72) gives the dispersion relation

$$m \tanh(mH) = \frac{g}{\omega^2}(k^2 + l^2), \tag{5.75}$$

which is similar to the dispersion relation for surface waves. Actually, the solution (5.74) is not an internal wave, but a surface gravity wave. To have internal waves, we need that $\omega^2 \leq N^2$. This is the next case to consider. We consider the general solution of equation (5.73), which is given by the equation

$$\hat{w}(z) \quad \sin[m(z+H)] \text{ with } m^2 \quad \frac{(N^2 - \omega^2)}{\omega^2}(k^2 + l^2), \tag{5.76}$$

which already satisfies the bottom boundary condition. If we substitute equation (5.76) into the free-surface boundary condition (5.72), we obtain the dispersion relation

$$N^2 - \omega^2 \quad gm \tanh(mH). \tag{5.77}$$

For a given value of the frequency ω, this dispersion relation gives a countable set of values for the modulus of the horizontal component $(k^2 + l^2)$ of the wavenumber, or for a given value of the modulus of the horizontal component of the wavenumber, we have a countable set of possible value for the frequency ω. For ω smaller or of the same order of the buoyancy frequency N, the free-surface displacement is small, and we can assume we have a rigid wall instead of a free surface, so equation (5.72) reduces to

$$\hat{w}(z) \quad 0 \text{ at } z \quad . \tag{5.78}$$

This boundary condition gives a dispersion relation of the form

$$\sin(mH) = 0 \tag{5.79}$$

or

$$\omega^2 \quad \frac{(k^2 + l^2)N^2H^2}{n^2 \;\; ^2 + (k^2 + l^2)H^2}, n \qquad , 2, 3, \dots , \tag{5.80}$$

which is close to the result given by the dispersion relation given by the free-surface boundary condition (5.77). The value of m for the case with a free-surface is slightly larger than the case with the rigid lid approximation.

If the ocean is perturbed with a spatial structure of one of the modes (a specific value of m for a given ω), then the subsequent behavior in time is described by equation (5.67), i. e., there is an oscillation with a particular frequency. Such a situation, however, is unlikely, so it is necessary to represent the initial structure in space as a *superposition* of modes (for a given ω, we have a countable set of values for $k^2 + l^2$). Then each of these will behave in time as found above, and so the solution can be constructed at all times by taking the appropriate superposition of modes.

5.2 Free Waves in a semi-infinite region.

The atmosphere does not have a definite upper boundary as does the ocean, so solutions of equation (5.73) will now be considered for the case of a semi-infinite domain $z > 0$. In this case there are two types of solutions, the first being typified by the case N *constant*. The only solutions of equation (5.73) that satisfy the condition at the ground z and remain bounded at infinity are sinusoidal, i. e., $=0$

$$\hat{w}(z) = \sin(mz), \tag{5.81}$$

where m has the same expression as the one given in equation (5.76). There is now no restriction on m, so according to the functional relation between m and ω given in equation (5.76), the frequency ω can have any value in the range $0 \leq \omega < N$, i. e., there is a continuous spectrum of solutions. Superposition of such solutions can be used to solve initial-value problems, and have the from of Fourier integrals.

When N varies with z, there is another type of solution possible, namely, one that satisfies the condition at the ground yet decays as $z \to \infty$. These are waveguide modes, and there are, in general, only a finite number possible. A simple example is provided by the case in which a region of depth H of uniform large buoyancy frequency N_1 underlies a semi-infinite region of uniform small buoyancy frequency N_2. The layer with buoyancy frequency N_1 has depth H and lies at $0 < z < H$ and the semi-infinite layer with buoyancy frequency N_2 lies at $z > H$. For $0 < \omega < N_2$, the solution in both layers has the form given by equation (5.81) with m m_1 in the first layer and m m_2 in the second layer. The wave frequency is constant across the interface of the two layers, which gives the relation

$$\frac{N_1^2}{m_1^2 + k^2 + l^2} \quad \frac{N_2^2}{m_2^2 + k^2 + l^2} \tag{5.82}$$

between the vertical wavenumbers m_1 and m_2. For this case, the spectrum is continuous and ω can assume any value between 0 and N_2. This is not true for the case when $N_2 < \omega < N_1$, when the frequency ω can assume only a finite set of values in the range $N_2 < \omega < N_1$. In this case, the solution of equation (5.73) for the first layer is given by the equation

$$\hat{w}(z) \quad \sin(m_1 z) \text{ where } m_1^2 \quad \frac{(N_1^2 - \omega^2)}{\omega^2}(k^2 + l^2), \tag{5.83}$$

and in the second layer we have solution given by the equation

$$\hat{w}(z) \quad \exp(-\gamma z) \text{ where } \gamma^2 \quad \frac{(\omega^2 - N_2^2)}{\omega^2}(k^2 + l^2). \tag{5.84}$$

At the intersection $z \quad H$ between the two layers, the perturbation pressure p' and the vertical velocity w should be continuous. Alternatively, this condition can be expressed in terms of the ratio

$$Z \quad \frac{p'}{\rho_0 w}, \tag{5.85}$$

which must be the same on both sides of the boundary. It is convenient to refer to Z as the "impedance". The condition that the impedance in both sides of the layers interface should be the same gives the possible values for ω (eigenvalues). This condition is expressed by the equation

$$\cot^2(m_1 H) = \frac{\omega^2 - N_2^2}{N_1^2 - \omega^2}. \tag{5.86}$$

The spectrum in terms of the wave frequency has a continuous part plus a discrete part, solution of equation (5.86). The modes $\hat{w}$ for $0 \quad {}_2$ are of sinusoidal shape in both layers, and for $N_2 \quad \omega \quad N_1$ the modes $\hat{w}(z)$ are sinusoidal in the first layer and decay exponentially in the second layer. Thus, to deduce how the perturbation will change with time from some initial state, it is necessary to represent this state as a superposition both of discrete waveguide modes and the continuous spectrum of sinusoidal modes. The relative amplitude of the different modes depends on the initial state.

6 Energetics of Internal Waves.

The energy equation for internal waves, under the assumption of small perturbations, incompressible and inviscid fluid and irrotational flow, can be obtained by multiplying

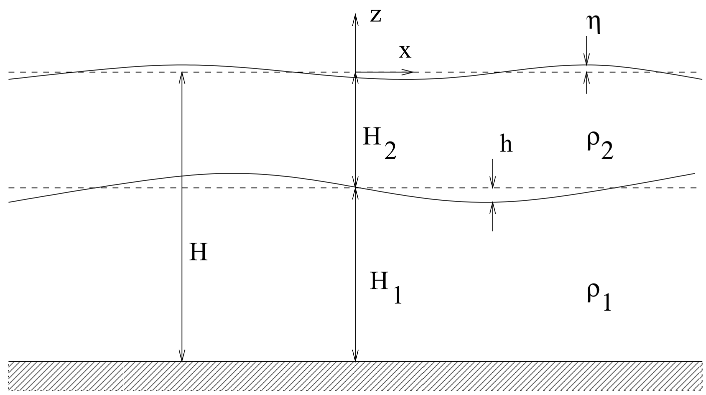

Figure 6: Two-layer fluid system with a free-surface.

equations (2.17), (2.18) and (2.19), respectively, by u, v and w, by multiplying equation (2.16) by $g^2\rho'/\bar{\rho}N^2$, and then adding the result and by taking into account the equation of continuity (2.7) and the definition of the buoyancy frequency. We obtain

$$\frac{\partial}{\partial t} \quad \frac{1}{2}\bar{\rho}\left(u^2+v^2+w^2\right)+\frac{1}{2}\frac{g\rho'}{\bar{\rho}N^2} \quad +\frac{\partial(p'u)}{\partial x}+\frac{\partial(p'v)}{\partial y}+\frac{\partial(p'w)}{\partial z} \qquad . \tag{6.87}$$

The term $\bar{\rho}(u^2+v^2+w^2)/2$ stands for the perturbation kinetic energy density. The term $(p'u, p'v, p'w)$ stands for the perturbation energy density flux and the term $\frac{1}{2}\frac{g^2 \;\; \prime 2}{\bar{\rho}N^2}$ stands for the perturbation potential energy density. The identification of this term with the perturbation potential energy density is less obvious, so it is helpful to consider the case of a two-layer fluid illustrated in the figure 6.

For this system, the potential energy is equal to

$$V \quad \int\int\int \rho g z dz dx dy \quad \int\int\left\{\frac{1}{2}\rho_2 g[\eta^2-(H_2-h)^2]+\frac{1}{2}\rho_1 g[(H_2-h)^2-H^2]\right\}dxdy$$

$$\int\int\left\{\frac{1}{2}\rho_2 g[\eta^2-(H_2-h)^2]+\frac{1}{2}\rho_1 g[(H_2-h)^2-H^2]\right\}dxdy \tag{6.88}$$

If we skip the constant terms in the equation above, we end up only with the potential energy associated with the energy due to the perturbation, which is equal to

$$\int\int\left\{\frac{1}{2}\rho_2 g\eta^2+\frac{1}{2}g(\rho_1-\rho_2)h^2\right\}dxdy, \tag{6.89}$$

and in a many layered system, each interface will contribute a term like

$$\int\int \frac{1}{2}(\rho_1 - \rho_2)h^2 dxdy, \tag{6.90}$$

and in the limit of a continuously stratified fluid this becomes

$$-\int\int\int \frac{1}{2} \quad \frac{\partial\bar{\rho}}{\partial z} \quad gh^2 dzdydx \quad \int\int\int \frac{1}{2}\bar{\rho}N^2h^2 dxdydz, \tag{6.91}$$

where h is the displacement of a fluid element from its equilibrium position. Since the density of a fluid element at its perturbed level $z + h$ is equal to the density $\bar{\rho}(z)$ at its equilibrium position, the perturbation density ρ' is given by

$$\rho' \quad \bar{\rho}(z) - \bar{\rho}(z + h) \approx - h\frac{\partial\bar{\rho}}{\partial z}, \tag{6.92}$$

and so the right hand side of equation (6.91) becomes

$$\int\int\int \frac{1}{2} \quad \frac{g^2\rho'^2}{\bar{\rho}N^2} \quad dxdydz. \tag{6.93}$$

The connection with (6.87) is now clear. For periodic waves in a medium with uniform properties, the integral over each wavelength is the same, and so the mean over a large volume becomes equal to the mean over one wavelength in the limit as the volume tends to infinity. Hence, it is useful to consider mean quantities rather than integrated quantities. The mean is being defined as the mean over a wavelength, and denoted by <>. The energy density of an internal wave is defined as the mean perturbation energy per unit volume. In other words, by

$$E \quad \frac{1}{2}\bar{\rho} \quad (u^2 + v^2 + w^2) > + \frac{1}{2}\frac{g^2 \quad '^2 >}{\bar{\rho}N^2} \tag{6.94}$$

When integrated over a large volume, equation (6.87) shows that the rate of change of energy over that volume is equal to the flux of energy across the sides. since this flux is also periodic, the average over a large plane area is approximately the same as the average over one wavelength, so it is convenient to consider the spatial mean for the fluxes as well. Thus the energy flux density vector $\overrightarrow{F'}$ is defined by the equation

$$\overrightarrow{F'} \qquad '(u', v', w') >, \tag{6.95}$$

<p

where (u', v', w') is the perturbation velocity vector.

7 Mountain Waves.

Internal waves in the atmosphere and ocean can be generated by a variety of mechanisms. Often the source region is approximately horizontal, so the vertical velocity component can effectively be specified on some horizontal surface, and the motion away from the source can be calculated from the equations of motion. We first consider the case in which air or water is moving with uniform horizontal speed over a succession of hills and valleys with elevation A_0 above the horizontal plane $z \quad 0$. Second, we consider air moving with uniform horizontal velocity over a single ridge (localized source) in an infinite atmosphere, and third, we consider a air or water moving with horizontal velocity over single ridge in a finite atmosphere or ocean (waveguide case). We assume that the atmosphere or ocean density stratification is such that the buoyancy frequency is constant.

7.1 Governing Equation.

We consider an air or water moving at a constant velocity $U(z)$ over a periodic or localized topography given by the equation

$$z \quad \zeta(x) \tag{7.96}$$

We assume the fluid incompressible and the flow irrotational. The fluid velocity vector (u, v) satisfies the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \qquad , \tag{7.97}$$

and the momentum equations

$$\rho \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \qquad - \frac{\partial p}{\partial x}, \tag{7.98}$$

$$\rho \quad \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \qquad - \frac{\partial p}{\partial z} - \rho g. \tag{7.99}$$

The fluid density has to satisfy the equation

$$\frac{1}{\rho}\frac{D\rho}{Dt} \tag{7.100}$$

Now we write the horizontal velocity in the form

$$u(x,z) = \ U(z) + u'(x,z), \tag{7.101}$$

We substitute equation (7.101) in the equations (7.97) to (7.100). We assume the velocities u' and w as small quantities, so we can linearize the resulting equations. The linearized form of the continuity equation is

$$\frac{\partial u'}{\partial x} + \frac{\partial w}{\partial z} \qquad , \tag{7.102}$$

and for the momentum equation, its linearized form is

$$\rho \quad \frac{\partial u'}{\partial t} + U\frac{\partial u}{\partial x} + w\frac{\partial U}{\partial z} \qquad - \frac{\partial p}{\partial x}, \tag{7.103}$$

$$\rho \quad \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} \qquad - \frac{\partial p}{\partial z} - \rho g. \tag{7.104}$$

For the density equation (7.100) we obtain

$$\frac{1}{\rho} \quad \frac{\partial \rho}{\partial t} + U\frac{\partial \rho}{\partial x} + u'\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} \tag{7.105}$$

We consider the wave motion as a result from the perturbation of the state of equilibrium, which is the state of rest. So the distribution of the density and pressure is the hydrostatic equilibrium distribution given by equation (2.12). When the motion develops, the pressure and density are given, respectively, by equations (2.13) and (2.14), and p' and ρ' are the pressure and density perturbations of the "background state". Now the momentum equation assume the form

$$\bar{\rho} \quad \frac{\partial u'}{\partial t} + U\frac{\partial u}{\partial x} + w\frac{\partial U}{\partial z} \qquad - \frac{\partial p'}{\partial x}, \tag{7.106}$$

$$\bar{\rho} \quad \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} \qquad - \frac{\partial p'}{\partial z} - \rho' g, \tag{7.107}$$

and the density equation assumes the form

$$\frac{1}{\bar{\rho}} \quad \frac{\partial \rho'}{\partial t} + U\frac{\partial \rho'}{\partial x} + w\frac{\partial \rho'}{\partial z} \qquad . \tag{7.108}$$

As we did in section 2, we would like to reduce the system of equations given by the equations (7.102), (7.106), (7.107) and (7.105) to a single partial differential equation describing the evolution of a flow quantity, like the vertical velocity, for example. To accomplish that we follow the steps below.

First, we apply the time derivative to the continuity equation (7.102) to obtain

$$\frac{\partial^2 u'}{\partial t \partial x} + \frac{\partial^2 w}{\partial t \partial z} \qquad . \tag{7.109}$$

Second, we take the x derivative of the equation (7.106) to obtain

$$\bar{\rho} \quad \frac{\partial^2 u'}{\partial x \partial t} + U\frac{\partial^2 u'}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial U}{\partial z} \qquad - \frac{\partial^2 p'}{\partial x^2} \tag{7.110}$$

Third, we eliminate the u' variable from the equation (7.110) above. To do so, we use equation (7.109) and the x derivative of the continuity equation (7.102). After the u' variable is eliminated, equation (7.110) assumes the form

$$\bar{\rho} \quad -\frac{\partial^2 w}{\partial z \partial t} - U\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial w}{\partial x}\frac{\partial U}{\partial z} \qquad - \frac{\partial^2 p'}{\partial x^2} \tag{7.111}$$

Fourth, we apply the operator $\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$ to the equation (7.107) to obtain

$$\bar{\rho} \quad \frac{\partial^2 w}{\partial t^2} + 2U\frac{\partial^2 w}{\partial x \partial t} + U^2\frac{\partial^2 w}{\partial x^2} \qquad -\frac{\partial^2 p'}{\partial t \partial z} - U\frac{\partial^2 p'}{\partial x \partial z} - g \quad \frac{\partial \rho'}{\partial t} + U\frac{\partial \rho'}{\partial x} \quad , \tag{7.112}$$

and with equation (7.105) we can eliminate ρ' from equation (7.112). The result is the equation

$$\bar{\rho} \quad \frac{\partial^2 w}{\partial t^2} + 2U\frac{\partial^2 w}{\partial x \partial t} + U^2\frac{\partial^2 w}{\partial x^2} \qquad -\frac{\partial^2 p'}{\partial t \partial z} - U\frac{\partial^2 p'}{\partial x \partial z} - gw\frac{\partial \bar{\rho}}{\partial z}, \tag{7.113}$$

Next, we apply the operator $\frac{\partial^2}{\partial x^2}$ to equation (7.113). Then, we eliminate p' from the resulting equation by using equation (7.111). We end up with the equation

$$\begin{aligned} \bar{\rho} \quad \frac{\partial^4 w}{\partial x^2 \partial t^2} + 2U\frac{\partial^4 w}{\partial x^3 \partial t} + U^2\frac{\partial^4 w}{\partial x^4} \qquad \frac{\partial}{\partial z} \quad \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \qquad -\bar{\rho}\frac{\partial^2 w}{\partial t \partial z} - \bar{\rho}U\frac{\partial^2 w}{\partial x \partial z} \\ + \bar{\rho}\frac{\partial w}{\partial x}\frac{\partial U}{\partial z} \quad + g\frac{\partial \bar{\rho}}{\partial z}\frac{\partial^2 w}{\partial x^2}, \end{aligned} \tag{7.114}$$

which is the desired partial differential equation in terms of the vertical velocity w. We can simplify the equation above. We can write it in the form

$$\frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \quad ^2 \quad \frac{\partial^2 w}{\partial x^2} + \frac{1}{\bar{\rho}}\frac{\partial}{\partial z} \quad \bar{\rho}\frac{\partial w}{\partial z} \qquad - \qquad \frac{\partial}{\partial t} + U\frac{\partial}{\partial x} \quad \frac{1}{\bar{\rho}}\frac{\partial}{\partial z} \quad \bar{\rho}\frac{\partial U}{\partial z}\frac{\partial w}{\partial x} \quad + N^2(z)\frac{\partial^2 w}{\partial x^2} \qquad , \tag{7.115}$$

where $N(z)$ is the buoyancy frequency defined according to equation (2.28). If we assume that w varies with z much more rapidly than $\bar{\rho}(z)$, then we can write

$$\frac{1}{\bar{\rho}}\frac{\partial}{\partial z} \quad \bar{\rho}\frac{\partial w}{\partial z} \quad \sim \frac{\partial^2 w}{\partial z^2} \tag{7.116}$$

and

$$\frac{1}{\bar{\rho}}\frac{\partial}{\partial z} \quad \bar{\rho}\frac{\partial U}{\partial z}\frac{\partial w}{\partial x} \quad \sim \frac{\partial}{\partial z} \quad \frac{\partial U}{\partial z}\frac{\partial w}{\partial x} \quad . \tag{7.117}$$

As a result, equation (7.115) simplifies to

$$\frac{\partial}{\partial t}+U\frac{\partial}{\partial x} \quad ^{2} \quad \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2} \quad - \quad \frac{\partial}{\partial t}+U\frac{\partial}{\partial x} \quad \frac{\partial}{\partial z} \quad \frac{\partial U}{\partial z}\frac{\partial w}{\partial x} \quad + N^2(z)\frac{\partial^2 w}{\partial x^2} \qquad , \tag{7.118}$$

We can simplify this equation further by assuming that the velocity U is constant. In this case we end up with an equation of the form

$$\frac{\partial}{\partial t}+U\frac{\partial}{\partial x} \quad ^{2} \quad \frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial z^2} \quad + N^2(z)\frac{\partial^2 w}{\partial x^2} \qquad , \tag{7.119}$$

Next, we discuss boundary conditions for the equations (7.114) to (7.119). Equations (7.114) to (7.119) are simpler versions of the equation (7.114). For the case of a finite or infinite atmosphere we need a boundary condition at the ground and a radiation condition as $z \to \infty$. The boundary condition at the ground is the condition of no flux trough the ground, given by the equation

$$\frac{D}{Dt}(z-\zeta(x)) \quad 0 \text{ or } \quad \frac{\partial}{\partial t}+(U+u')\frac{\partial}{\partial x}+w\frac{\partial}{\partial z} \quad (z-\zeta(x)) \quad 0 \text{ at } z \quad \zeta(x) \tag{7.120}$$

We linearize the boundary condition on the ground located at $z \quad \zeta(x)$. We expand the terms in equation (7.120) with respect to z , and we obtain the linear boundary condition

$$w(x,z,t)= \; U\frac{\partial \zeta}{\partial x} \text{ on } z \tag{7.121}$$

For an infinite atmosphere we need a radiation condition, which ensures that the energy flux is away from the ground. In other words, energy is radiated away from the ground by the internal waves generated by the topography. For the case of a finite atmosphere, we need a boundary condition on the top part of the atmosphere. We postpone the discussion of the radiation condition or the top boundary condition when we consider specific ground topographies (periodic, single ridge, etc.).

7.2 Periodic Mountain.

We consider a topography of sinusoidal mountains. In this case, $\zeta(x)$ is given by the equation

$$\zeta(x)= \ A_0 \sin(kx) \tag{7.122}$$

where k is the wavenumber of the topography. The boundary condition (7.121) at the ground for this case assumes the form

$$w(x,z,t)= \ A_0 U k \cos(kx) \text{ at } z \tag{7.123}$$

In this example, we assume a constant buoyancy frequency N for the entire atmosphere. Under such condition, we can assume a solution for the steady state regime of the form

$$w(x,z,t)= \ A\cos(kx+mz), \tag{7.124}$$

where k is the topography wavenumber, since the solution (7.124) has to satisfy the boundary condition at the ground, given by the equation (7.123). m is the wavenumber in the vertical direction. After we substitute the solution (7.124) into the boundary condition (7.123), we obtain that

$$A \qquad A_0 U k. \tag{7.125}$$

The governing equation for this problem (constant buoyancy frequency) is given by equation (7.119). If we substitute the solution (7.124) into equation (7.119), we obtain the an expression for the vertical wavenumber, which follows

$$m^2 \qquad \frac{N^2}{U^2} - k^2 \tag{7.126}$$

According to the equation above, if $\frac{N}{U} > k$, them m is real and we obtain waves which propagates through the atmosphere. If $\frac{N}{U} \qquad k$, them m is imaginary and we have a solution which decays exponentially away from the ground.

7.2.1 Case $\frac{N}{U} > k$.

There are two solutions for equation (7.126) (plus or minus sign of the square root of the right hand side of equation (7.126)), and to decide which one we should use to represent energy being radiated away from the ground we are going to consider the energy flux in the vertical direction. According to equation (6.95), the average energy flux in the vertical direction is

$$F'_z \qquad {}'w > \tag{7.127}$$

$<p$

where p' can be obtained from equation (7.111) in terms of the vertical perturbation velocity w. For the vertical velocity given by equation (7.124), we have

$$p' \quad \bar{\rho} U A \frac{m}{k} \cos(kx + mz) \tag{7.128}$$

Now, we compute the average vertical energy flux, and we obtain the expression

$$F'_z \quad \frac{1}{2} \bar{\rho} U A^2 \frac{m}{k}, \tag{7.129}$$

and since we need a positive vertical average energy flux for the energy to be radiated away from the ground, we chose m as given by the equation

$$m \qquad \sqrt{\frac{N^2}{U^2} - k^2}. \tag{7.130}$$

$=$

Since energy is being radiated away from the ground, there is drag exerted by the topography due to the generation of the internal waves. The magnitude of the drag force per unit area is equal to the rate τ per unit area at which horizontal momentum is transfered vertically by the waves. This is given by the equation

$$\begin{aligned} \tau \quad & -\bar{\rho}\bar{u}\bar{w} \quad F'_z/U \\ & \frac{1}{2}\bar{\rho} A^2 \frac{m}{k} \end{aligned} \tag{7.131}$$

For the vertical wavenumber m equal to zero, we have the horizontal *cut-off wavenumber* k_c, given by the equation

$$k_c \quad \frac{N}{U}. \tag{7.132}$$

This wavenumber divides the two types of solutions ($N/U > k$ and $N/U \quad k$), and corresponds to a wavelength $2\pi/k_c$ equal to the horizontal distance traveled by a fluid particle in one buoyancy period. From equation (7.126) we have that

$$\frac{N}{U} \quad k^2 + m^2 \quad k_c)^2.$$

Thus, the angle ϕ' between wave crests and the vertical changes according to the equation

$$\cos\phi' \quad \frac{k}{k_c} \quad \frac{Uk}{N} \tag{7.133}$$

The angle ϕ' given by the equation above is illustrated in the figure 7.

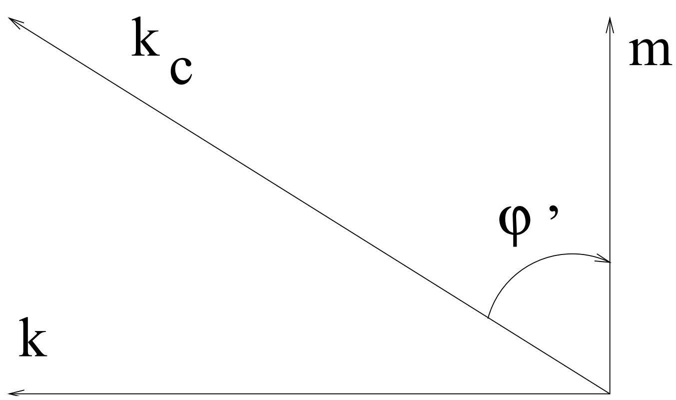

Figure 7: Angle ϕ'.

7.2.2 Case $\frac{N}{U}$. k

In this case the wavenumber m is a pure imaginary number, and the solution given by equation (7.124) assume the form

$$w(x,z,t) = \; A\Re\{\exp(-\gamma z + ikx)\} \tag{7.134}$$

where γ is given by the equation

$$\gamma^2 \quad k^2 - \frac{N^2}{U^2}. \tag{7.135}$$

This solution does not represents a propagating wave. It decays exponentially as we move away from the topography. There is no energy radiated away from the ground and there is no drag force exerted by the topography. To see this, we consider the expression for the pressure in this case, which is given by the equation

$$p' \quad \Re\{-i\bar{\rho}\gamma U \frac{A}{k} \exp(-\gamma z + ikx)\} \tag{7.136}$$

This is out of phase with the vertical velocity, i.e., is zero when w is a maximum or a minimum, and is a maximum or a minimum when w is zero. Thus, the rate F_z' of doing work by the topography over the atmosphere is zero.

7.3 Localized Topography in an Infinite Atmosphere.

In this section we consider the steady state disturbance in an infinite atmosphere due to a stream with a horizontal constant speed U passing over a localized topography $z \quad \zeta(x)$. We assume that the buoyancy frequency N is constant. For a reference frame fixed to the ground, the governing equation (7.119) assumes the form

$$U^2 \frac{\partial^2}{\partial x^2} \quad \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \quad + N^2 \frac{\partial^2 w}{\partial x^2} \qquad . \tag{7.137}$$

In this reference frame we will obtain a steady disturbance downstream to the localized topography. So, in this reference frame we have zero wave frequency and zero phase speed. If we consider a reference frame moving with the stream (constant speed U), we will see the localized topography moving with speed $-U$, and we will see waves with phase speed $-U$ matching the speed on the topography (steady disturbance downstream to the topography in the fixed reference frame). Here we consider the second reference frame, since in this reference frame we can easily speak in terms of phase and group velocity, which makes more simple the discussion of the radiation condition for this case. The governing equation in the moving reference frame is given by equation (2.30) without the y component. In other words,

$$\frac{\partial^2}{\partial t^2} \quad \frac{\partial^2 w}{\partial (x')^2} + \frac{\partial^2 w}{\partial z^2} \quad + N^2 \frac{\partial^2 w}{\partial (x')^2} \qquad , \tag{7.138}$$

where x' is the horizontal axis in the moving reference frame. The horizontal axis in the fixed and moving frame are related according to the equation

$$x \qquad x' + Ut. \tag{7.139}$$

The boundary condition on the ground, given by the equation (7.121), for this reference frame assumes the form

$$w(x', z, t) = \; U \frac{\partial \zeta}{\partial x}(x' + Ut) \text{ at } z \qquad . \tag{7.140}$$

To solve the boundary value problem given by equations (7.138) and (7.140) with the appropriate radiation condition, we consider the Fourier transform pair in the x' variable, given by the equations

$$\hat{f}(k) = \; \int_{-\infty}^{+\infty} f(x) \exp(-ikx') dx' \tag{7.141}$$

and

$$f(x) = \; \frac{1}{2} \int_{-\infty}^{+\infty} \hat{f}(k) \exp(ikx') dk. \tag{7.142}$$

We apply the Fourier transform to the governing equation (7.138) and to the boundary condition (7.140). The governing equation (7.138) in the wavenumber domain assumes the form

$$\frac{\partial^2}{\partial t^2} \quad -k^2 + \frac{\partial^2 \hat{w}}{\partial z^2} \quad - k^2 N^2 \hat{w} \qquad , \tag{7.143}$$

and the boundary condition (7.140) assumes the form

$$\hat{w}(k,z,t) = \; ikU\hat{\zeta}(k)\exp(ikUt) \tag{7.144}$$

Next, we consider a time dependence of the form

$$\hat{w}(k,z,t) = \; \hat{w}'(k,z)\exp(-i\omega t) \tag{7.145}$$

If we substitute equation (7.145) into the governing equation (7.143) we end up with the governing equation for $\hat{w}'$, which follows:

$$\omega^2 \frac{\partial^2 \hat{w}'}{\partial z^2} + (N^2k^2 - \omega^2 k^2)\hat{w}' \tag{7.146}$$

Solutions of the equation (7.146) are in the form

$$\hat{w}'(k,z) = \; A\exp(-im(k)z), \tag{7.147}$$

where $m(k)$ can be obtained in terms of k and ω by substituting equation (7.147) into equation (7.146). Its expression is given by the equation

$$m^2 \qquad \frac{k^2(N^2 - \omega^2)}{\omega^2}, \tag{7.148}$$

and which branch we are going to choose from the two possible branches given by equation (7.148) will be decided to satisfy the radiation condition that energy is transported by the internal waves away from the ground. Equation (7.148) can be rewritten as a dispersion relation, which follows

$$\omega \qquad \frac{kN}{\sqrt{m^2 + k^2}} \tag{7.149}$$

From the dispersion relation above we can obtain the group velocity vector. The group velocity is the speed with which energy is propagated by the internal waves, it is the gradient of the wave frequency with respect to the wavenumber, then

$$\overrightarrow{C}_g \qquad \frac{m^2 N}{(m^2+k^2)^{3/2}}, -\frac{mkN}{(m^2+k^2)^{3/2}} \quad . \tag{7.150}$$

Next, we are going to discuss the radiation condition for this problem and to decide about which branch of $m(k,\omega)$ given by equation (7.148) we should use. The appropriate radiation condition for this problem is that energy should be radiated away from the ground. This implies that the vertical component of the group velocity vector should be positive. In other words,

$$-\frac{mkN}{(m^2+k^2)^{3/2}} > 0. \tag{7.151}$$

The inequality (7.151) implies that for $k > 0$ we need m to be negative and for $k \quad 0$ we need m to be positive. As a result, we chose the appropriate branch of m given by equation (7.148) as follows:

- For $k \geq 0$ we have that

$$m(k,N/U)= \begin{cases} -\left[\left(\frac{U}{N}\right)^2 - k^2\right]^{1/2} & \text{if } |k| \leq \frac{N}{U} \\ -i\left[k^2 - \left(\frac{N}{U}\right)\right]^{1/2} & \text{if } |k| > \frac{N}{U} \end{cases}, \tag{7.152}$$

- For $k< 0$ we have that

$$m(k,N/U)= \begin{cases} \left[\left(\frac{U}{N}\right)^2 - k^2\right]^{1/2} & \text{if } |k| \leq \frac{N}{U} \\ -i\left[k^2 - \left(\frac{N}{U}\right)\right]^{1/2} & \text{if } |k| > \frac{N}{U} \end{cases}, \tag{7.153}$$

where we used in equation (7.148) the fact that

$$\frac{\omega}{k} \quad -U \text{ or } k \quad -\frac{\omega}{U}, \tag{7.154}$$

which is the dopler effect related to the change from a fixed to a moving reference frame. According to equations (7.117), (7.150) and (7.154), the horizontal component of the group velocity vector C_{gx} can be written as

$$C_{gx} \quad -U\frac{m^2}{|\overrightarrow{k}|^2}, \tag{7.155}$$

which implies a negative value for the horizontal component of the group velocity. From this equation we also realize that the horizontal component of the group velocity is in the same direction as the horizontal component of the phase speed, but smaller magnitude. This implies that we see waves downstream the localized topography. If the magnitude of the horizontal component of the group speed was larger than the horizontal component of the phase speed, the wave disturbance would have appear upstream of the localized topography. From equations (7.145) and (7.147), we can write the expression for $\hat{w}(k, z, t)$ in the form

$$w(x', z, t)= \ A \exp(-im(k, N/U)z + ikUt) \tag{7.156}$$

To obtain the solution, we substitute equation (7.156) in the boundary condition (7.144). We obtain

$$A \quad ikU\hat{\zeta}(k). \tag{7.157}$$

Now the expression for $\hat{w}(k, z, t)$ can be obtained with the help of equations (7.156), (7.152), (7.153) and (7.157), and if we apply the inverse Fourier transform to the resulting equation, we obtain

$$\begin{aligned} w(x', z, t)= \ -\frac{U}{\ } \Bigg\{ \int_0^{N/U} ik\hat{\zeta}(k) \sin(\sqrt{\quad \frac{N}{U} \quad ^2} - k^2 z + kUt + kx')dk \\ + \int_{N/U}^{\infty} ik\hat{\zeta}(k) \exp(-\sqrt{k^2 - \quad \frac{N}{U} \quad ^2}\, z) \sin(+kUt + kx')dk \Bigg\} \end{aligned} \tag{7.158}$$

In terms of the fixed reference frame, the vertical velocity w is given by the equation

$$w(x,z) = -\frac{U}{}\left\{\int_0^{N/U} ik\hat{\zeta}(k)\sin(\sqrt{\frac{N}{U}^{2} - k^2}\,z + kx)dk \right.$$
$$\left. + \int_{N/U}^{\infty} ik\hat{\zeta}(k)\exp(-\sqrt{k^2 - \frac{N}{U}^{2}}\,z)\sin(+kx)dk\right\} \tag{7.159}$$

since $x' \quad x - Ut$. Next, we consider an example of a localized topography, the "witch of Agnesi", for which $\zeta(x)$ (fixed reference frame) is given by the equation

$$\zeta(x) = \frac{A_0}{1 + (x/b)^2}, \tag{7.160}$$

and its Fourier transform is given by the equation

$$\hat{\zeta}(k) = A_0 b \exp(-|kb|). \tag{7.161}$$

For this particular example, the vertical velocity is given by

$$w(x,z) = -UA_0 b\left\{\int_0^{N/U} k\exp(-b|k|)\sin(\sqrt{\frac{N}{U}^{2} - k^2}\,z + kx)dk \right.$$
$$\left. + \int_{N/U}^{\infty} k\exp(-b|k| - \sqrt{k^2 - \frac{N}{U}^{2}}\,z)\sin(+kx)dk\right\} \tag{7.162}$$

To obtain a picture of the flow over the topography, it is more appropriate to consider stream lines instead of the vertical velocity w. The relationship between the stream function $\psi(x,z)$ and the vertical velocity $w(x,z)$ is given by the equation

$$w(x,z) = \frac{D\psi}{Dt} \quad U\frac{\partial\psi}{\partial x}. \tag{7.163}$$

According to the equations (7.162) and (7.163), the stream function $\psi(x,z)$ for this problem is given by the equation

$$\psi(x,z)= A_0 b \left\{ \int_0^{N/U} \exp(-b|k|) \sin(\sqrt{\frac{N}{U}^{2} - k^2 z + kx)dk} \right.$$
$$\left. + \int_{N/U}^{\infty} \exp(-b|k| - \sqrt{k^2 - \frac{N}{U}^{2}} z) \sin(+kx)dk \right\} \tag{7.164}$$

By inspecting equation (7.164), if the buoyancy frequency N is zero, the stream function ψ has a simple expression given by the equation

$$\psi(x,z)= \frac{UA_0}{1+[x/(b+z)]^2}, \tag{7.165}$$

and we realize that no wave disturbance is generated downstream the topography, as illustrated in figure (8). On the other hand, if the buoyancy frequency N is non-zero, there always waves generated downstream the topography as illustrated in figure (9).

7.4 Localized Topography in a Finite Atmosphere.

In this section we consider the steady state disturbance in a finite atmosphere due to the stream with horizontal constant speed U passing over a localized topography $z \quad \zeta(x)$. We assume the buoyancy frequency N constant along the atmosphere. For a reference frame fixed on the ground, we have the governing equation (7.137) for the flow vertical velocity w. For this governing equation we have the boundary condition at the ground, given by equation (7.121), and we also consider a boundary condition at $z \quad h$, which is the top of the finite atmosphere. This boundary condition is given by the equation

$$w(x,z) \quad 0 \text{ at } z \quad h. \tag{7.166}$$

In this case no energy is radiated in the vertical direction. Instead enery is transported downstream of the topography or it is not radiated at all, as we will see from the discussion and results that will follow along this section. We consider in this section a reference system moving with the stream (constant velocity U with respect to the ground), as we did in the previous section. In this reference system, the governing

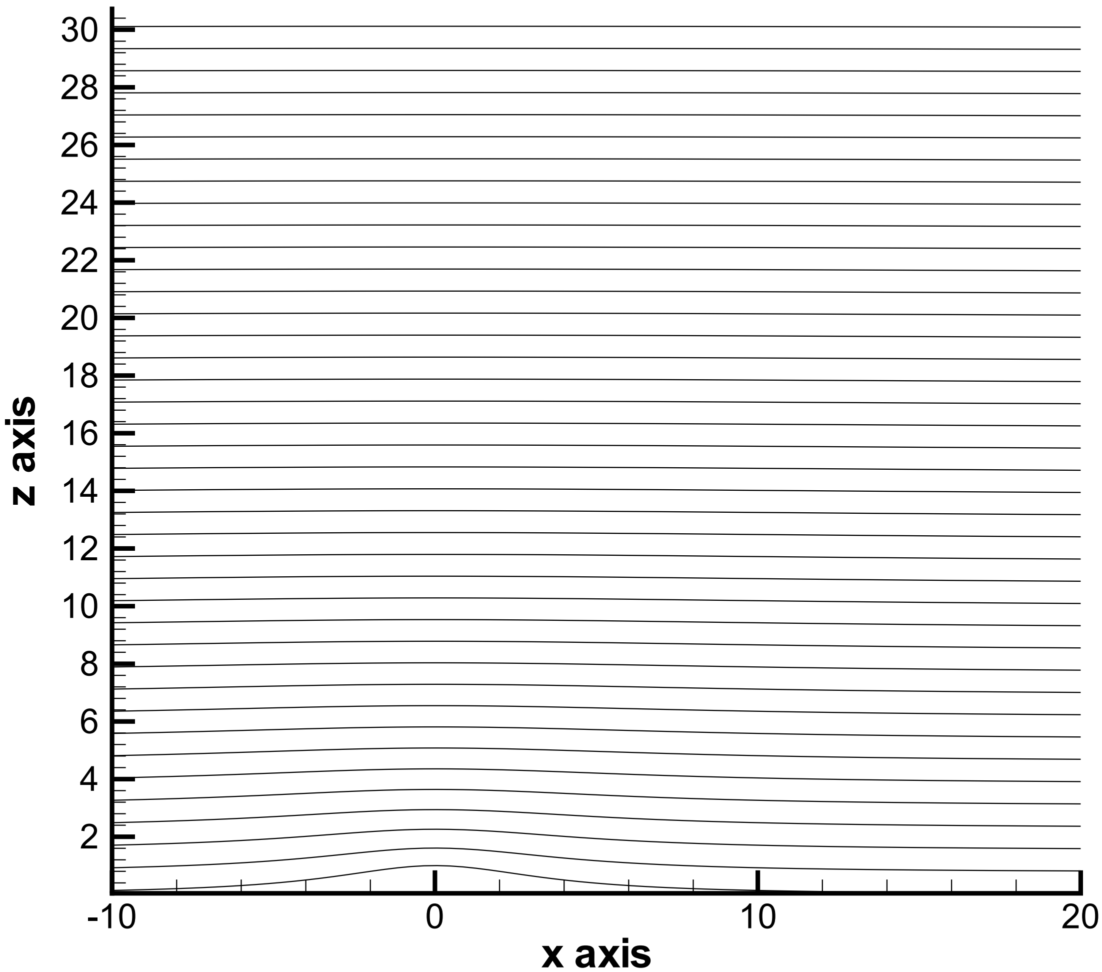

Figure 8: Stream lines for the case of zero value for the buoyancy frequency N. A_0 and b . =4

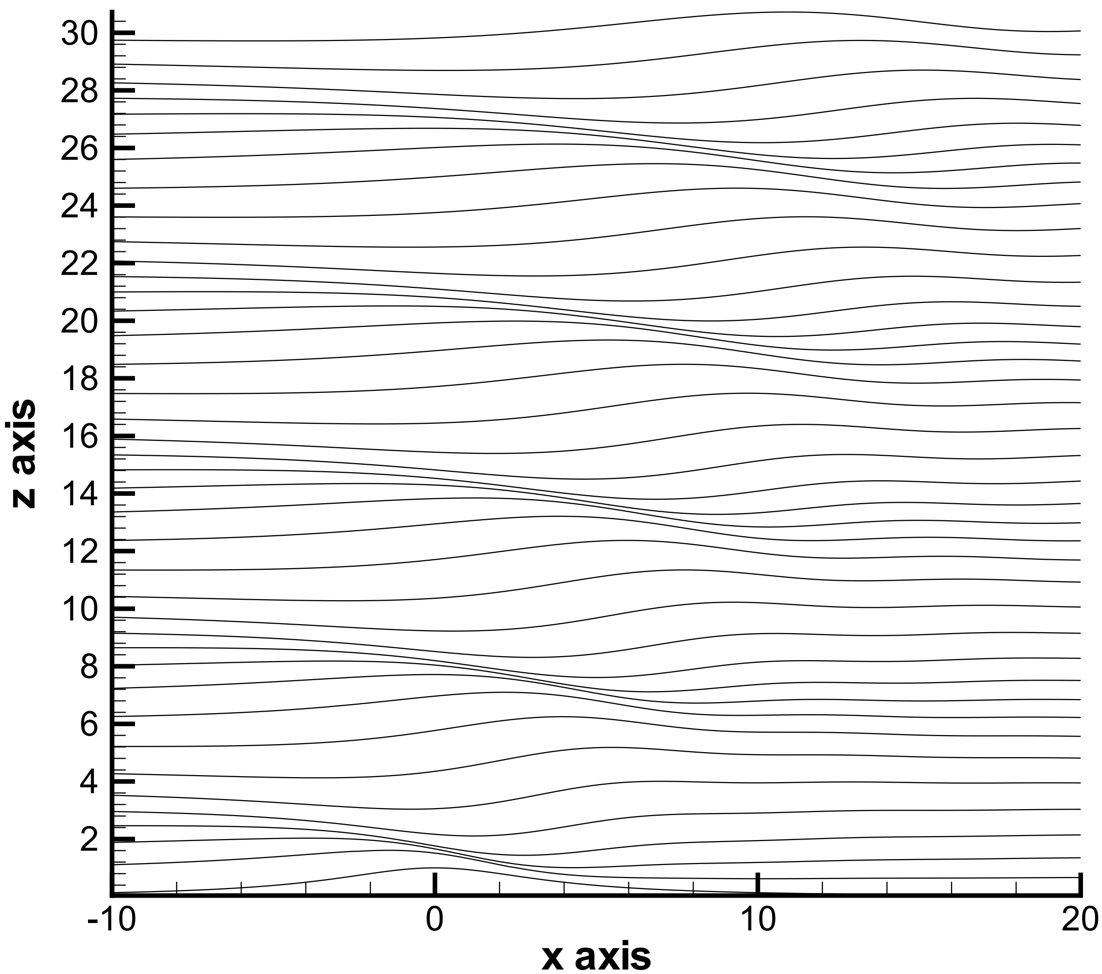

Figure 9: Stream lines for the case of non-zero value for the buoyancy frequency N. N/U , A_0 1 and b . =1 =4

equation for the vertical flow velocity w is given by equation (2.30). At the ground, we have the boundary condition given by equation (7.140), and at the top of the atmosphere we consider the boundary condition (7.166). As in the previous section, we consider the Fourier transform pair given by equations (7.141) and (7.142). For $\hat{w}(k, z, t)$ (Fourier transform of $w(x', z, t)$), we consider the time dependence

$$\hat{w}(k, z, t) = \hat{w}'(k, z) \exp(-i\omega t). \tag{7.167}$$

The governing equation for $\hat{w}'(k, z)$ is given by equation (7.146) and it has also to satisfy the boundary condition (7.144) and the Fourier transform of the boundary condition (7.166). This implies that we should have $\omega \quad -kU$. We assume a general solution for $\hat{w}'$ of the form

$$\hat{w}' \quad B \sinh(m(h - z)), \tag{7.168}$$

which already satisfies the Fourier transform of equation (7.166). If we substitute equation (7.167) into the ground boundary condition given by equation (7.144), we obtain

$$B \quad \frac{ikU\hat{\zeta}(k)}{\sinh(mh)}, \tag{7.169}$$

and if we substitute equation (7.168) into the governing equation (7.146), we obtain for the vertical wavenumber m the expression

$$m \quad \frac{k}{\omega}(\omega^2 - N^2)^{1/2} \quad k^2 - (N/U)^2)^{1/2}, \tag{7.170}$$

or we can obtain a dispersion relation, which follows

$$\omega \quad \frac{kN}{\sqrt{k^2 + m^2}}. \tag{7.171}$$

This dispersion relation will be necessary to obtain the group velocity of the wave disturbances generated by the topography, which will be necessary to discuss how to deform the integration contour of the inverse Fourier transform of $\hat{w}(k,z,t)$. By substituting equation (7.169) into equation (7.168), we obtain for $\hat{w}(k,z,t)$ the expression

$$\hat{w}(k,z,t)= ikU\hat{\zeta}(k)\frac{\sinh(m(z-h))}{\sinh(mh)}\exp(ikUt), \tag{7.172}$$

and its inverse Fourier transform gives the expression for $w(x',z,t)$, which follows:

$$w(x',z,t)= \frac{U}{2}\int_{-\infty}^{\infty} ik\hat{\zeta}(k)\frac{\sinh(m(z-h))}{\sinh(mh)}\exp(ik(Ut+x'))dk \tag{7.173}$$

This inverse Fourier has closed form solution, which is basically the sum of the residue of part of the poles of the integrand in equation (7.173). To obtain these poles, we use

$$\sinh z \quad z\prod_{l=1}^{\infty} \quad 1+\frac{z^2}{l^2 \;\; ^2} \tag{7.174}$$

Therefore, we can write

$$\frac{\sinh(m(h-z))}{\sinh(mh)} \quad \frac{(h-z)\prod_{l=1}^{\infty}\left(1+\frac{m^2(h-z)^2}{l^2 \;\; ^2}\right)}{h\prod_{l=1}^{\infty}\left(1+\frac{(mh)^2}{l^2 \;\; ^2}\right)}, \tag{7.175}$$

which implies that we have poles in the k complex plane for

$$k \quad \pm\left\{ \quad \frac{N}{U} \quad ^2 - l^2\left(\frac{}{h}\right)^2\right\}^{1/2} \text{ with } l \quad\quad ,2,\ldots \tag{7.176}$$

These are first order poles which are usually pure imaginary numbers, but we can have real poles for non-zero buoyancy frequency values if $\frac{N}{U} > \frac{l\pi}{h}$ for some values of l (smaller values of l). The real poles are associated with waves generated by the presence of the localized topography. These are called Lee waves. If $\frac{N}{U}$ $\frac{l\pi}{h}$ for $l \geq 1$, there is no real poles, which implies no waves associated with the localized topography. In this case we have just a local evanescent wavefield close to the localized topography.

To evaluate the integral in equation (7.173), we consider a closed contour, which is the original integration contour along the real axis plus a semi-circle of infinite radius lying in the upper half part of the complex k plane for $(Ut + x') > 0$, or a semi-circle of infinite radius lying in the lower half part of the complex k plane for $(Ut + x') \quad 0$. We consider a deformation also of the real axis to take into account the real poles that we may have. We have to decide if we deform the real axis to pass above or below the real poles. This decision is associated if we have waves downstream or upstream of the localized topography, and to carry it out, we need to compute the group velocity, which is given by

$$C_g \quad \frac{m^2 N}{(m^2 + k^2)^{3/2}} \quad \frac{\omega}{k} \frac{m^2}{|\overrightarrow{k}|}, \tag{7.177}$$

where ω/k is the phase velocity of the wave following the topography, which has value $-U$. Then we have

$$C_g \quad -U \frac{m^2}{|\overrightarrow{k}|} \quad -U \quad 0, \tag{7.178}$$

which implies group velocity in the same direction of the phase velocity, but with magnitude less than the phase velocity. This implies that the waves are behind the localized topography (downstream). If the group speed was larger than the phase velocity, the waves would be ahead of the topography (upstream). As a result, we consider the contribution of the real poles only for the case $(Ut + x') > 0$. Then, we deform the real axis to pass below the real poles that we might have. The integration contours are illustrated in figure 10.

The contribution of the integral along the part of the contour that lies at the infinite in the complex k plane is zero. Therefore, the only contribution comes from the poles inside the closed contour illustrated in the figure 10. Now, the expression for the vertical velocity $w(x', z, t)$ can be written as follows:

- Case $(Ut + x') > 0$. We assume that the first L poles are real, and the poles for $l > L$ are pure imaginary. We have that

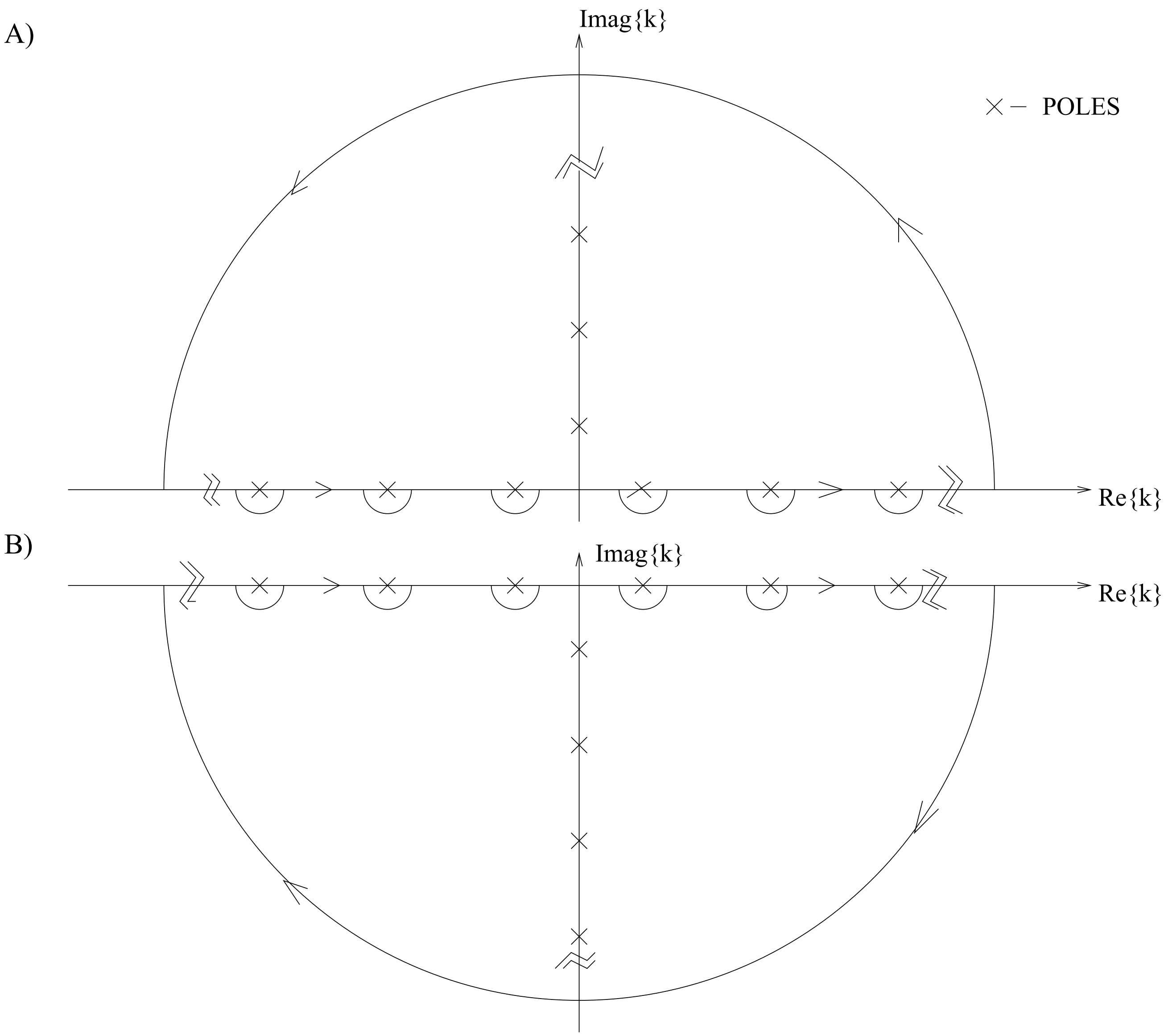

Figure 10: (A) - Deformed integration contour for the case $(Ut+x') > 0$, (B) - Deformed integration contour for the case $(Ut + x') \quad 0$.

$$
\begin{aligned}
w(x', z, t) = & \sum_{j=1}^{L} -\frac{i}{2}\text{Res} \quad ikU\hat{\zeta}(k)\frac{\sinh(m(k, N/U)(h-z))}{\sinh(m(k, N/U)h)}e^{(ik(Ut+x'))} \quad |_{k=\sqrt{(N/U)^2-(j\pi/h)^2}} \\
& + \sum_{j=1}^{L} -\frac{i}{2}\text{Res} \quad ikU\hat{\zeta}(k)\frac{\sinh(m(k, N/U)(h-z))}{\sinh(m(k, N/U)h)}e^{(ik(Ut+x'))} \quad |_{k=-\sqrt{(N/U)^2-(j\pi/h)^2}} \\
& + \sum_{j=L+1}^{\infty} -i\text{Res} \quad ikU\hat{\zeta}(k)\frac{\sinh(m(k, N/U)(h-z))}{\sinh(m(k, N/U)h)}e^{(ik(Ut+x'))} \quad |_{k=i\sqrt{(j\pi/h)^2-(N/U)^2}}
\end{aligned}
\tag{7.179}
$$

where Res stands for residue and

$$m(k, N/U) = \quad \sqrt{k^2 - (N/U)^2} \tag{7.180}$$

- Case $(Ut + x') \quad 0$. We assume only poles that are pure imaginary numbers lying in the lower half part of the complex k plane. We have that

$$w(x', z, t) = \sum_{j=L+1}^{\infty} i\text{Res} \quad ikU\hat{\zeta}(k)\frac{\sinh(m(k, N/U)(h-z))}{\sinh(m(k, N/U)h)}e^{(ik(Ut+x'))} \quad |_{k=-i\sqrt{(j\pi/h)^2-(N/U)^2}} \tag{7.181}$$

Since, the minimum value of l is one, the critical speed for a given value of the buoyancy frequency N is $U \quad \frac{Nh}{}$. For current values $U > \frac{Nh}{}$, there is no wave disturbance downstream of the localized topography.

To illustrate the flow for this problem, we change from the moving reference frame to the fixed reference frame $(x' + Ut \quad x)$, and we consider the flow stream function $\psi(x, z)$ defined in the previous section in terms of the vertical velocity $w(x, z)$ according to the equation (7.163). Therefore, the stream function $\psi(x, z)$ is given by the equation

$$\psi(x, z) = \frac{1}{2}\int_{-\infty}^{\infty}\hat{\zeta}(k)\frac{\sinh(m(z-h))}{\sinh(mh)}\exp(ikx)dk, \tag{7.182}$$

and its evaluation in terms of the residues at the poles $\sqrt{(N/U)^2 - (j\pi/h)^2}$ follows:

- Case $x > 0$. We assume that the first L poles are real, and the poles for $l > L$ are pure imaginary. We have that

$$\begin{aligned}\psi(x,z)= &\sum_{j=1}^{L} -\frac{i}{2}\mathrm{Res}\quad \hat{\zeta}(k)\frac{\sinh(m(k,N/U)(h-z))}{\sinh(m(k,N/U)h)}\exp(ikx)\quad |_{k=\sqrt{(N/U)^2-(j\pi/h)^2}} \\ &+\sum_{j=1}^{L} -\frac{i}{2}\mathrm{Res}\quad \hat{\zeta}(k)\frac{\sinh(m(k,N/U)(h-z))}{\sinh(m(k,N/U)h)}\exp(ikx)\quad |_{k=-\sqrt{(N/U)^2-(j\pi/h)^2}} \\ &+\sum_{j=L+1}^{\infty} -i\mathrm{Res}\quad \hat{\zeta}(k)\frac{\sinh(m(k,N/U)(h-z))}{\sinh(m(k,N/U)h)}\exp(ikx)\quad |_{k=i\sqrt{(j\pi/h)^2-(N/U)^2}}\end{aligned} \tag{7.183}$$

- Case $x< 0$. We assume only poles that are pure imaginary numbers lying in the lower half part of the complex k plane. We have that

$$\psi(x,z)= \sum_{j=L+1}^{\infty} i\mathrm{Res}\quad \hat{\zeta}(k)\frac{\sinh(m(k,N/U)(h-z))}{\sinh(m(k,N/U)h)}\exp(ikx)\quad |_{k=i\sqrt{(j\pi/h)^2-(N/U)^2}} \tag{7.184}$$

The equations (7.183) and (7.184) can be written in a simple way in terms of the quantities θ_j, γ_j and α_j defined in the appendix A.

- Case $x > 0$, where the first L poles are assumed real numbers, and the other poles are in the upper part of the complex k plane.

$$\psi(x,z)= \sum_{j=1}^{L} -\beta_j\theta_j\sin(\alpha_j x) - \sum_{j=L+1}^{\infty}\beta_j\gamma_j\exp(-\alpha_j x), \tag{7.185}$$

where β_j is defined as

$$\beta_j \quad \hat{\zeta}(\pm\alpha_j) \text{ and } \beta_j \quad \hat{\zeta}(\pm i\alpha_j), \tag{7.186}$$

since the chosen topography has Fourier transform even with respect to the real and imaginary axis of the complex k plane.

- Case x 0, where we consider the poles above the real axis in the complex k plane.

$$\psi(x,z) = -\sum_{j=L+1}^{\infty} \beta_j \gamma_j \exp(\alpha_j x), \tag{7.187}$$

We chose for $\zeta(x)$ the same topography we considered in the previous section. The stream lines for this flow are illustrated in figure 11, 12 and 13.

For stream speeds that approach the critical values $\frac{Nh}{l\pi}$ from below, the group velocity C_g for the l-th lee wave approaches the stream speed. Therefore, according to the linear theory, no energy is radiated away from the source by this lee wave. If l , no energy is radiated away from the source, since for values of U just below this critical value we have only one lee wave. In this situation, non-linear effects became important and waves are generated upstream, as discussed qualitatively in the next section.

8 Upstream Influence.

A Residue of the poles for the finite atmosphere problem.

Here we give the expression for the residue of the integrand in equation (7.182), which is used to compute the flow streamlines illustrated in the figures above. First, we label the wavenumbers as α_j, as follows:

- For $N/U > j \;\; /h$ we have,

$$\pm\alpha_j \quad \pm\sqrt{(N/U)^2 - (j\pi/h)^2} \tag{A.188}$$

- For N/U $/h$ we have, $<j$

$$\pm i\alpha_j \quad \pm i\sqrt{(j\pi/h)^2 - (N/U)^2} \tag{A.189}$$

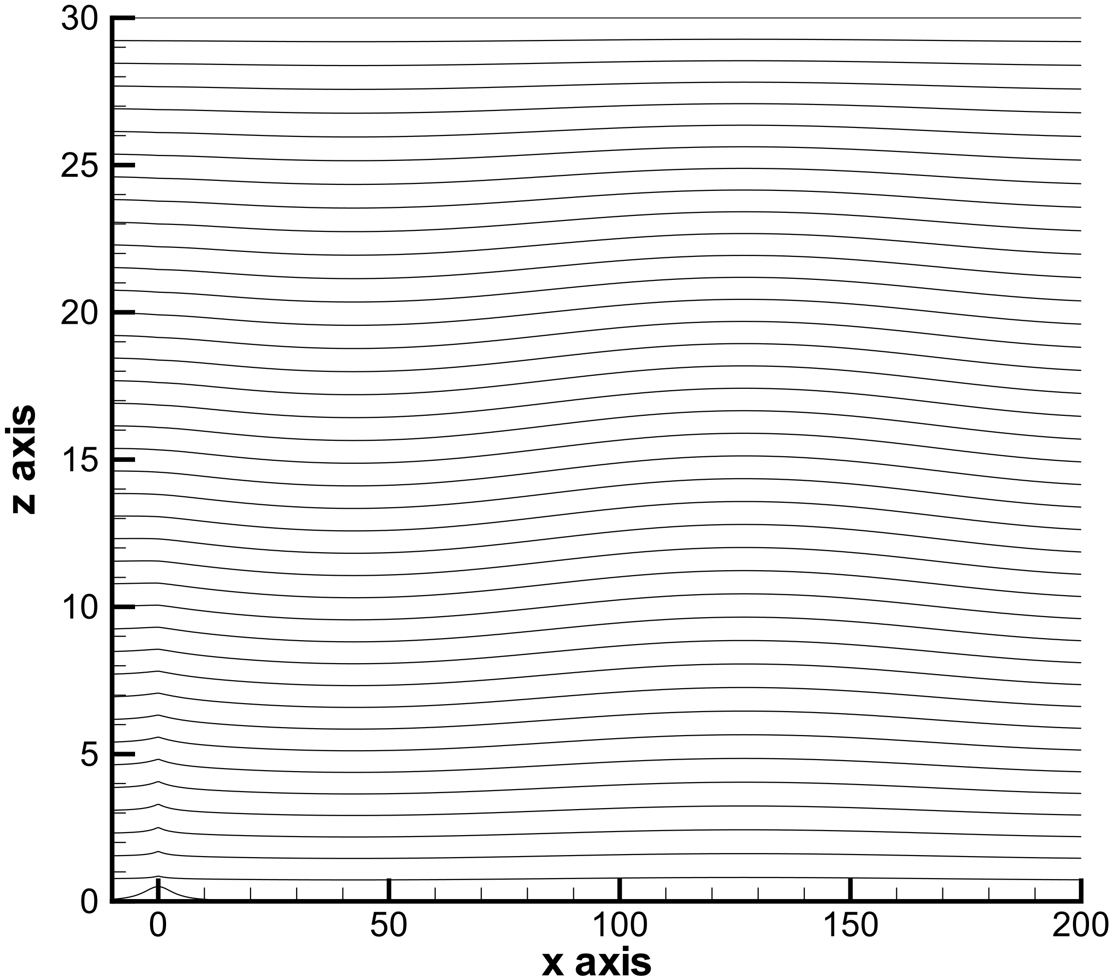

Figure 11: Stream functions for the "Witch of Agnasi" topography in a finite atmosphere with U/N and A_0 $/2$. We expect to see a single lee wave for this value $=9$ of N/U. $=1$

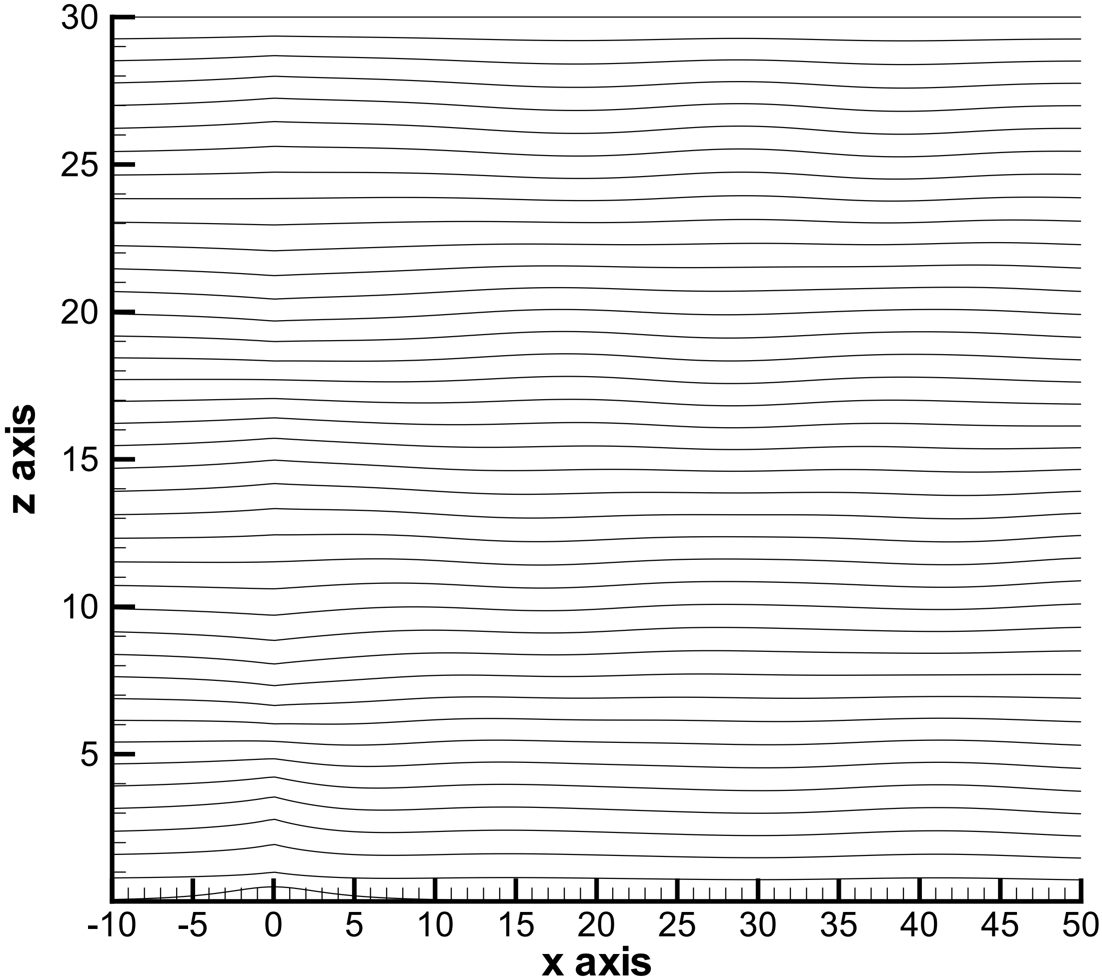

Figure 12: Stream functions for the "Witch of Agnasi" topography in a finite atmosphere with U/N 2 and A_0 1/2. We expect to see a superposition of four lee waves for this value of N/U.

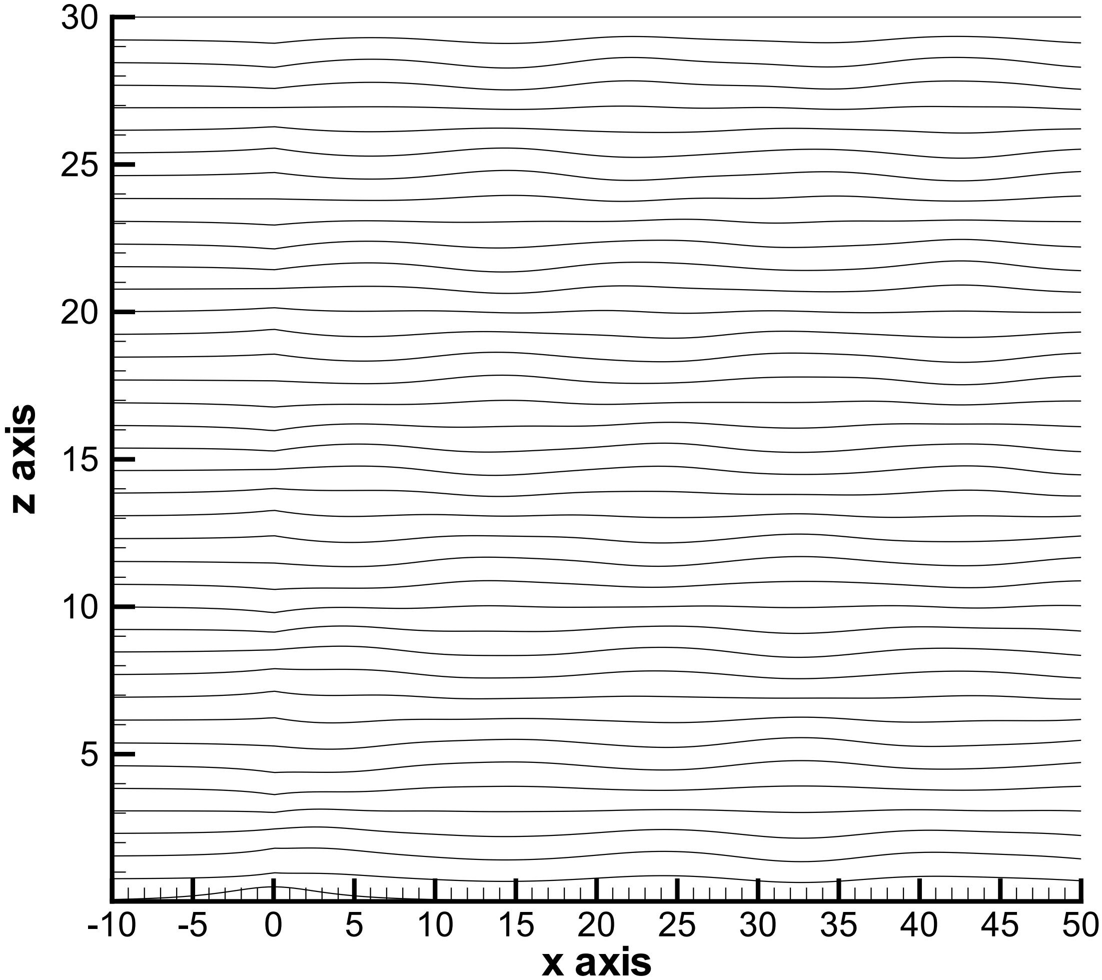

Figure 13: Stream functions for the "Witch of Agnasi" topography in a finite atmosphere with U/N 1 and A_0 /2. We expect to see a superposition of nine lee wave for this =1 value of N/U.

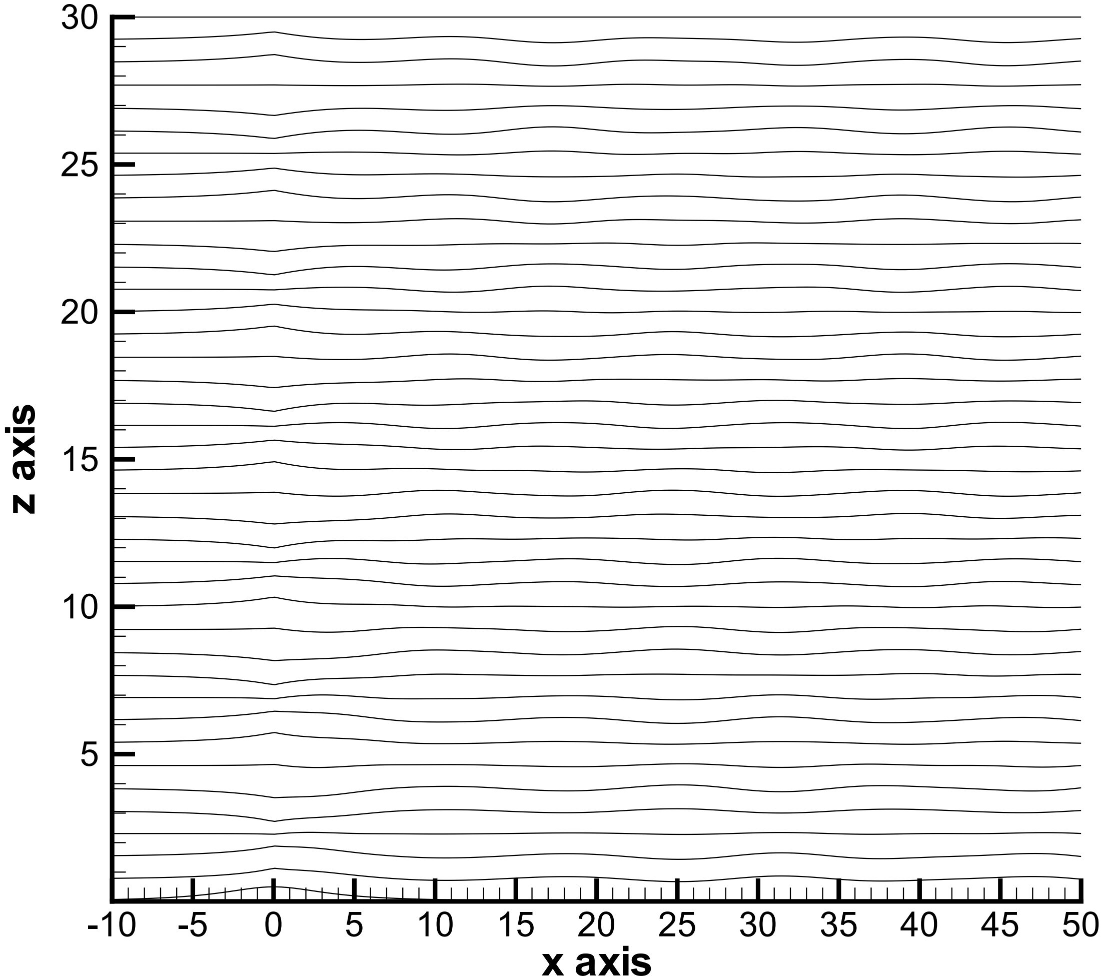

Figure 14: Stream functions for the "Witch of Agnasi" topography in a finite atmosphere with U/N $/4$ and A_0 $/2$. We expect to see a superposition of twelve $l=3$ wave for $=1$ this value of N/U.

Next, we give the expression for the residues in equations (7.183) and (7.184).

- Case $x \geq 0$. We consider in this case the real poles for which $j \quad 1, \ldots, L$. We first give the residue at the real poles.

$$\text{Res} \quad \hat{\zeta}(k) \frac{\sinh(m(k, N/U)(h-z))}{\sinh(m(k, N/U)h)} \exp(ikx)dk \quad |_{k=\pm\alpha_j} \quad \pm\hat{\zeta}(\pm\alpha_j)\theta_j, \tag{A.190}$$

where θ_j is given by the equation:

$$\theta_j \quad \frac{(j\pi/h^2)\sin[j\pi(h-z)/h]}{\cos(j\pi)\sqrt{(N/U)^2 - (j\pi/h)^2}} \text{ for } j \quad , \ldots, L \tag{A.191}$$

The poles in the upper part of the complex k plane give the residue which follows:

$$\text{Res} \quad \hat{\zeta}(k) \frac{\sinh(m(k, N/U)(h-z))}{\sinh(m(k, N/U)h)} \exp(ikx)dk \quad |_{k=i\alpha_j} \quad \hat{\zeta}(i\alpha_j)(i\gamma_j), \tag{A.192}$$

where γ_j is given by the equation:

$$\gamma_j \quad \frac{(j\pi/h^2)\sin[j\pi(h-z)/h]}{\cos(j\pi)\sqrt{(j\pi/h)^2 - (N/U)^2}} \text{ for } j \quad L+1, \ldots \tag{A.193}$$

- Case $x < 0$. We consider for this case the poles in the lower part of the complex k plane. The residue at these poles follows:

$$\text{Res} \quad \hat{\zeta}(k) \frac{\sinh(m(k, N/U)(h-z))}{\sinh(m(k, N/U)h)} \exp(ikx)dk \quad |_{k=-i\alpha_j} \quad \hat{\zeta}(-i\alpha_j)(-i\gamma_j) \tag{A.194}$$