2.092/2.093 — Finite Element Analysis of Solids & Fluids I	Fall '09

Lecture 6 - Finite Element Solution Process

Prof. K. J. Bathe — MIT OpenCourseWare

In the last lecture, we used the principle of virtual displacements to obtain the following equations:

$$\boldsymbol{K}\boldsymbol{U} = \boldsymbol{R} \tag{1}$$

$$\begin{aligned}
\boldsymbol{K} &= \sum_m \boldsymbol{K}^{(m)} \quad ; \quad \boldsymbol{K}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)T}\boldsymbol{C}^{(m)}\boldsymbol{B}^{(m)} dV^{(m)} \\
\boldsymbol{R} &= \boldsymbol{R}_B + \boldsymbol{R}_S \\
\boldsymbol{R}_B &= \sum_m \boldsymbol{R}_B^{(m)} \quad ; \quad \boldsymbol{R}_B^{(m)} = \int_{V^{(m)}} \boldsymbol{H}^{(m)T}\boldsymbol{f}^{B(m)} dV^{(m)} \\
\boldsymbol{R}_S &= \sum_m \boldsymbol{R}_S^{(m)} \quad ; \quad \boldsymbol{R}_S^{(m)} = \sum_i \int_{S_f^{i(m)}} \boldsymbol{H}^{S_f^{i(m)}T}\boldsymbol{f}^{S_f^{i(m)}} dS_f^{i(m)}
\end{aligned}$$

$$\boldsymbol{u}^{(m)} = \boldsymbol{H}^{(m)}\boldsymbol{U} \tag{2}$$

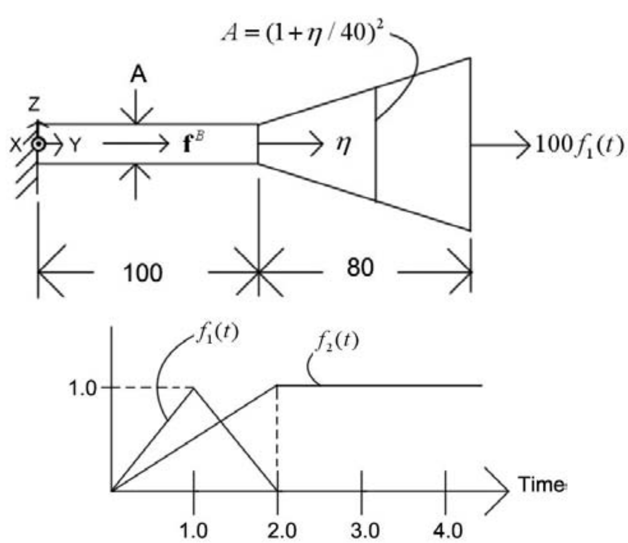
$$\downarrow$$

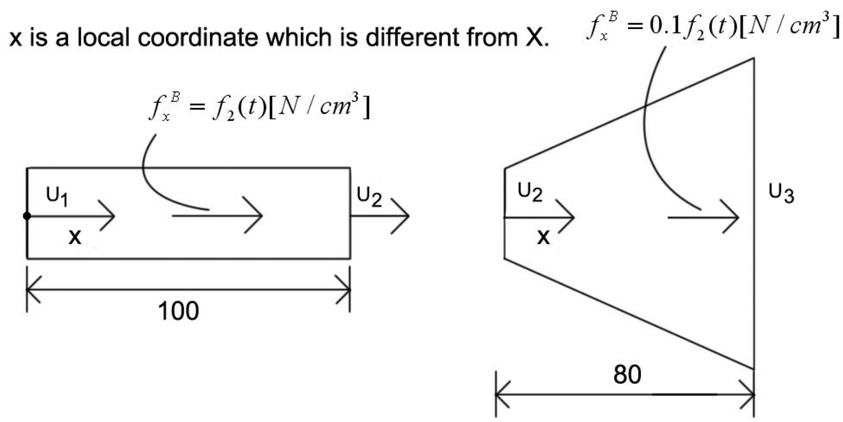
$$\boldsymbol{\varepsilon}^{(m)} = \boldsymbol{B}^{(m)}\boldsymbol{U} \tag{3}$$

Note that the dimension of $\boldsymbol{u}^{(m)}$ is in general not the same as the dimension of $\boldsymbol{\varepsilon}^{(m)}$.

Example: Static Analysis

Reading assignment: Example 4.5

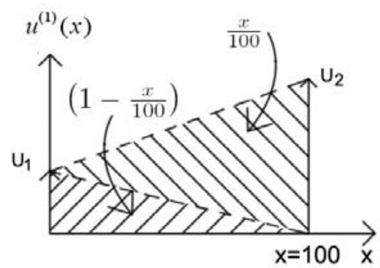




Assume:

i. Plane sections remain plane
ii. Static analysis → no vibrations/no transient response
iii. One-dimensional problem; hence, only one degree of freedom per node

Elements 1 and 2 are compatible because they use the same U_2. Next, use a linear interpolation function.

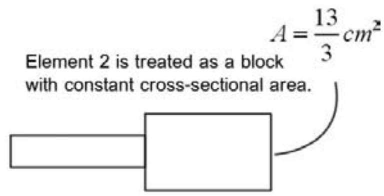


$$u^{(1)}(x) = \underbrace{\begin{bmatrix} \left(1-\frac{x}{100}\right) & \frac{x}{100} & 0 \end{bmatrix}}_{\boldsymbol{H}^{(1)}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \quad ; \quad u^{(2)}(x) = \underbrace{\begin{bmatrix} 0 & \left(1-\frac{x}{80}\right) & \frac{x}{80} \end{bmatrix}}_{\boldsymbol{H}^{(2)}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\varepsilon^{(1)}(x) = \underbrace{\begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix}}_{\boldsymbol{B}^{(1)}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \quad ; \quad \varepsilon^{(2)}(x) = \underbrace{\begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix}}_{\boldsymbol{B}^{(2)}} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{K} &= E \cdot 1 \cdot \int_0^{100} \begin{bmatrix} -\frac{1}{100} \\ \frac{1}{100} \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{100} & \frac{1}{100} & 0 \end{bmatrix} dx + E \int_0^{80} \left(1+\frac{x}{40}\right)^2 \begin{bmatrix} 0 \\ -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{80} & \frac{1}{80} \end{bmatrix} dx \\ &= \frac{E}{100} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{13E}{3 \cdot 80} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{aligned}$$

The "equivalent cross-sectional area" of element 2 is $A = \frac{13}{3}\ cm^2$. This equivalent area must lie between the areas of the end faces $A = 1$ and $A = 9$.



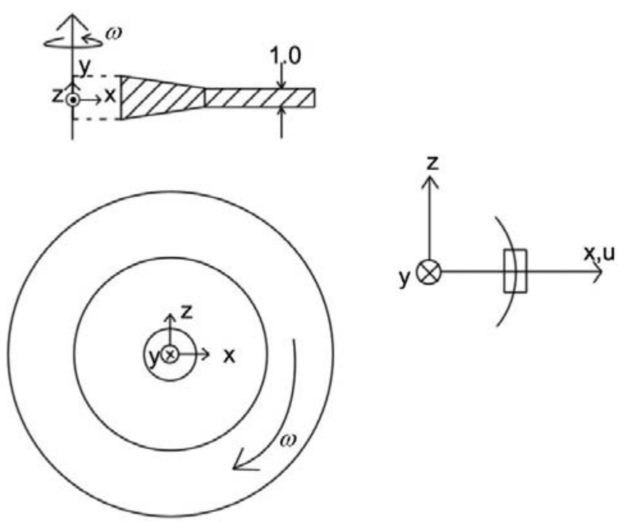
$$\boldsymbol{K} = \frac{E}{240}\begin{bmatrix} 2.4 & -2.4 & 0 \\ -2.4 & 15.4 & -13 \\ 0 & -13 & 13 \end{bmatrix}$$

We note:

- Diagonal terms **must** be positive. If the diagonal terms are zero or negative, then the system is unstable physically. A positive diagonal implies that the degree of freedom has stiffness at that node.
- $\boldsymbol{K}$ is symmetric.
- $\boldsymbol{K}$ is singular if rigid body motions are possible. To be able to solve the problem, all rigid body modes must be removed by adequately constraining the structure. i.e. $\boldsymbol{K}$ is reduced by applying boundary conditions to the nodes.

The $\boldsymbol{K}$ used to solve for $\boldsymbol{U}$ is, then, positive definite ($\det \boldsymbol{K} > 0$). This ensures that the elastic strain energy is positive and nonzero for any displacement field $\boldsymbol{U}$. In the analysis, each element is in equilibrium under its nodal forces, and each node is in equilibrium when summing element forces and external loads.

Homework Problem 2



$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{u}{x} \end{bmatrix}$$

ε_{zz} is frequently called the "hoop strain", $\varepsilon_{\theta\theta}$.

$$\varepsilon_{zz} = \frac{2\pi(u+x) - 2\pi x}{2\pi x} = \frac{u}{x}$$

$$\boldsymbol{C} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}$$

$$\boldsymbol{f}^B = \rho\omega^2 R \left[N/\text{cm}^3\right] \quad ; \quad R = x$$

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