2.092/2.093 — Finite Element Analysis of Solids & Fluids I Fall '09

Lecture 12 - FEA of Heat Transfer/Incompressible Fluid Flow

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Reminder: Quiz #1, Oct. 29. Closed book, 4 pages of notes.
Reading assigment: Section 7.4.2

We recall the principle of virtual temperatures.

$$\int_V \bar{\boldsymbol{\theta}}'^T \boldsymbol{k}\boldsymbol{\theta}' dV = \int_V \bar{\theta}^T q^B dV + \int_{S_q} \bar{\theta}^T q^S dS$$

$$\theta^{(m)} = \boldsymbol{H}^{(m)}\boldsymbol{\theta} \quad ; \quad \bar{\theta}^{(m)} = \boldsymbol{H}^{(m)}\bar{\boldsymbol{\theta}}$$

$$\boldsymbol{\theta}'^{(m)} = \boldsymbol{B}^{(m)}\boldsymbol{\theta} \quad ; \quad \bar{\boldsymbol{\theta}}'^{(m)} = \boldsymbol{B}^{(m)}\bar{\boldsymbol{\theta}}$$

$$\left(\sum_m \boldsymbol{K}^{(m)}\right)\boldsymbol{\theta} = \sum_m \left(\boldsymbol{Q}_B^{(m)} + \boldsymbol{Q}_S^{(m)}\right)$$

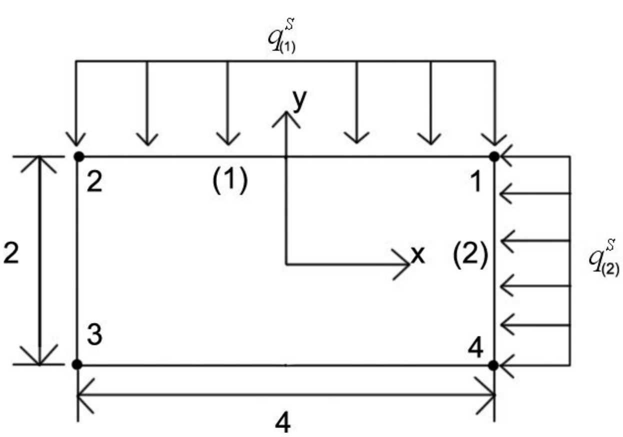
$$\boldsymbol{K}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)T}\boldsymbol{k}^{(m)}\boldsymbol{B}^{(m)} dV^{(m)}$$

$$\boldsymbol{Q}_B^{(m)} = \int_{V^{(m)}} \boldsymbol{H}^{(m)T} q^{B(m)} dV^{(m)}$$

$$\boldsymbol{Q}_S^{(m)} = \int_{S_q^{(m)}} \boldsymbol{H}^{S^{(m)}T} q^{S(m)} dS^{(m)} \qquad \text{(a)}$$

In (a), we may have to integrate over two or more surfaces.

Example



$$\boldsymbol{H} = \left[\ \tfrac{1}{4}\left(1+\tfrac{x}{2}\right)(1+y) \quad \tfrac{1}{4}\left(1-\tfrac{x}{2}\right)(1+y) \quad \tfrac{1}{4}\left(1-\tfrac{x}{2}\right)(1-y) \quad \tfrac{1}{4}\left(1+\tfrac{x}{2}\right)(1-y)\ \right]$$

$$\boldsymbol{B} = \begin{bmatrix} h_{1,x} & h_{2,x} & h_{3,x} & h_{4,x} \\ h_{1,y} & h_{2,y} & h_{3,y} & h_{4,y} \end{bmatrix}$$

$$\boldsymbol{H}^S\big|_{(1)} = \boldsymbol{H}^S\big|_{y=+1} = \left[\ \tfrac{1}{2}\left(1+\tfrac{x}{2}\right) \quad \tfrac{1}{2}\left(1-\tfrac{x}{2}\right) \quad 0 \quad 0\ \right]$$

$$\boldsymbol{H}^S\big|_{(2)} = \boldsymbol{H}\big|_{x=+2} = \left[\begin{array}{cccc} \frac{1}{2}(1+y) & 0 & 0 & \frac{1}{2}(1-y) \end{array}\right]$$

For our example, (a) means

$$\boldsymbol{Q}_S = \int_{S(1)} \boldsymbol{H}^S \big|_{(1)} q^S_{(1)} dS + \int_{S(2)} \boldsymbol{H}^S \big|_{(2)} q^S_{(2)} dS$$

$$q^S = h\left(\theta^e - \theta^S\right) = h\left(\theta^e - \boldsymbol{H}^S\boldsymbol{\theta}\right)$$

If θ^e varies, we can use $\theta^e = \boldsymbol{H}^S\boldsymbol{\theta}^e$.

In transient solutions,

$$q^B = \tilde{q}^B - \rho c\dot{\theta} \quad ; \quad \dot{\theta} = \boldsymbol{H}^{(m)}\dot{\boldsymbol{\theta}}$$

and $\tilde{q}^B$ no longer includes the rate at which heat is stored within the material. Putting this all together, we have:

$$\boldsymbol{C}\dot{\boldsymbol{\theta}} + \boldsymbol{K}\boldsymbol{\theta} = \tilde{\boldsymbol{Q}}_B + \boldsymbol{Q}_S$$

We also need the initial condition ${}^0\boldsymbol{\theta} = \boldsymbol{\theta}\,\big|_{t=0}$ to solve.

In all cases (linear, nonlinear, transient solutions) we solve the following

$$\boldsymbol{Q}_{int} = \boldsymbol{Q}_{applied}$$

This is analogous to $\boldsymbol{F} = \boldsymbol{R}$ in stress analysis. We further define:

$$\boldsymbol{Q}_{int} = \sum_m \boldsymbol{Q}_{int}^{(m)} \quad ; \quad \boldsymbol{Q}_{int}^{(m)} = \int_{V^{(m)}} \boldsymbol{B}^{(m)T}\boldsymbol{q}^{(m)}dV^{(m)}$$

$$\boldsymbol{q}^{(m)} = \boldsymbol{k}^{(m)}\boldsymbol{B}^{(m)}\boldsymbol{\theta}$$

By solving these equations, we have

- Finite element equilibrium
- Nodal point equilibrium

For any time t, the following should be satisfied:

$$\boxed{{}^t\boldsymbol{Q}_{int} = {}^t\boldsymbol{Q}_{applied}}$$

This equation is generally solved by Newton-Raphson iterations (see previous lecture notes).

Incompressible Fluid Flow

$$\rho v_{i,j}v_j = \tau_{ij,j} + f_i^B \tag{1}$$

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} \tag{2}$$

We define the velocity strain tensor e_{ij} as

$$e_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \tag{3}$$

Continuity in an incompressible fluid requires that

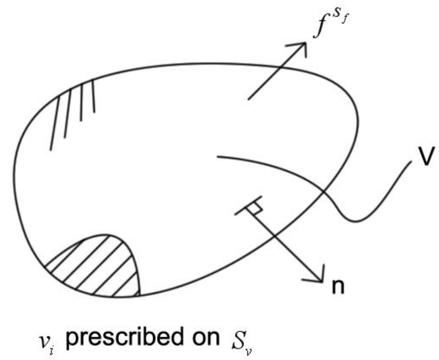
$$v_{i,i} = 0 \quad ; \quad \nabla\cdot\boldsymbol{v} = 0 \tag{4}$$

The unknowns are v_i for $i = 1, 2, 3$ and p.

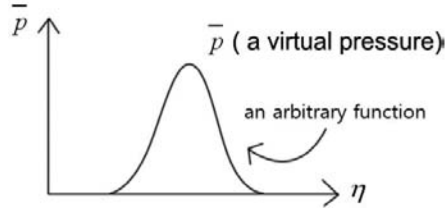
Principle of Virtual Velocities

$$\underbrace{\int_V \bar{v}_i \left(\rho v_{i,j} v_j\right) dV + \int_V \bar{e}_{ij}\tau_{ij} dV}_{\boldsymbol{F}} = \int_V \bar{v}_i f_i^B dV + \int_{S_f} \bar{v}_i^{S_f} f_i^{S_f} dS \tag{A}$$

$$\bar{e}_{ij} = \frac{1}{2}\left(\bar{v}_{i,j} + \bar{v}_{j,i}\right)$$



$$\int_V \bar{p} v_{i,i} dV = 0 \tag{B}$$



Next, we interpolate:

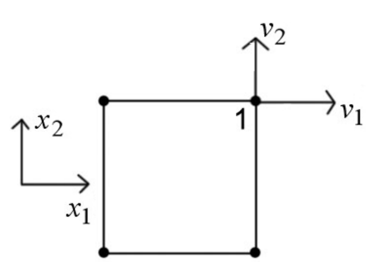
$$v_i = \boldsymbol{H}\boldsymbol{v} \quad ; \quad p = \boldsymbol{H}_p \boldsymbol{p}$$

Consider a 2D element:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix} \boldsymbol{v}$$

where v_i^j means the nodal velocity in the i direction at the node j. For a 4-node element,

$$\boldsymbol{v} = \begin{bmatrix} v_1^1 \\ v_1^2 \\ v_1^3 \\ v_1^4 \\ v_2^1 \\ v_2^2 \\ v_2^3 \\ v_2^4 \end{bmatrix}$$



$$p(x,y) = \boldsymbol{H}_p \boldsymbol{p} \quad ; \quad x = x_1, y = x_2$$

We need to "wisely" choose the $\boldsymbol{H}_p$ matrix for a given $\boldsymbol{H}$. Why? Assume the medium is slightly compressible. Then,

$$p = -\kappa \left(\nabla \cdot \boldsymbol{v}\right)$$

where κ is the given bulk modulus, which is very large. If the medium is a solid, then $\boldsymbol{v}$ = displacements. If the medium is a fluid, then we use $\dot{p}$ and $\boldsymbol{v}$ = velocity. Since $\boldsymbol{v}$ is approximated, error occurs. Even a small error in $\nabla \cdot \boldsymbol{v}$ will get magnified by the bulk modulus κ and will cause large errors in p.

The interpolation gives the expression

$$\boldsymbol{F}\left(\boldsymbol{v}, \boldsymbol{p}\right) = \boldsymbol{R}$$

which is what we should solve. Thus, in incompressible fluid flow analysis, the same finite element conditions hold as in stress and heat transfer analysis.

There are many more interesting and important points:

- The coefficient matrix is generally non-symmetric.
- High Reynolds number flows need special interpolation schemes.
- The number of elements is generally very large (will be discussed in 2.094).

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