**2.094 — Finite Element Analysis of Solids and Fluids** Fall '08

# Lecture 11 - Deformation, strain and stress tensors

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Reading: Ch. 6

We stated that we use

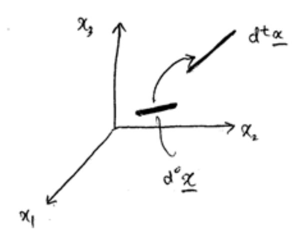
$$\int_{^tV} {}^t\tau_{ij}\ \delta_t e_{ij}\ d\,^tV = \int_{^0V} {}^t_0S_{ij}\ \delta\ {}^t_0\epsilon_{ij}\ d\,^0V = {}^t\mathcal{R} \tag{11.1}$$

**The deformation gradient** We use ${}^tx_i = {}^0x_i + {}^tu_i$

$$ {}^t_0\boldsymbol{X} = \begin{bmatrix} \frac{\partial\,^tx_1}{\partial\,^0x_1} & \frac{\partial\,^tx_1}{\partial\,^0x_2} & \frac{\partial\,^tx_1}{\partial\,^0x_3} \\ \frac{\partial\,^tx_2}{\partial\,^0x_1} & \frac{\partial\,^tx_2}{\partial\,^0x_2} & \frac{\partial\,^tx_2}{\partial\,^0x_3} \\ \frac{\partial\,^tx_3}{\partial\,^0x_1} & \frac{\partial\,^tx_3}{\partial\,^0x_2} & \frac{\partial\,^tx_3}{\partial\,^0x_3} \end{bmatrix} \tag{11.2}$$

$$d^t\boldsymbol{x} = \begin{bmatrix} d^tx_1 \\ d^tx_2 \\ d^tx_3 \end{bmatrix} \tag{11.3}$$

$$d^0\boldsymbol{x} = \begin{bmatrix} d^0x_1 \\ d^0x_2 \\ d^0x_3 \end{bmatrix} \tag{11.4}$$




Implies that

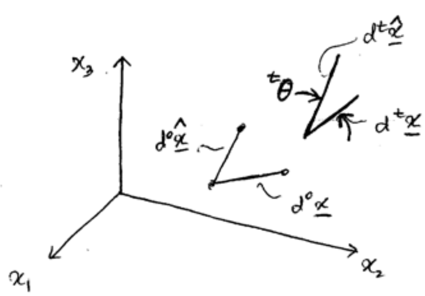
$$\boxed{d^t\boldsymbol{x} = {}^t_0\boldsymbol{X}\, d^0\boldsymbol{x}} \tag{11.5}$$

(${}^t_0\boldsymbol{X}$ is frequently denoted by ${}^t_0\boldsymbol{F}$ or simply $\boldsymbol{F}$, but we use $\boldsymbol{F}$ for force vector)

We will also use the right Cauchy-Green deformation tensor

$$\boxed{{}^t_0\boldsymbol{C} = {}^t_0\boldsymbol{X}^T\, {}^t_0\boldsymbol{X}} \tag{11.6}$$

**Some applications**



The stretch of a fiber (${}^t\lambda$):

$$\left({}^t\lambda\right)^2 = \frac{d\,{}^t\boldsymbol{x}^T d\,{}^t\boldsymbol{x}}{d\,{}^0\boldsymbol{x}^T d\,{}^0\boldsymbol{x}} = \left(\frac{d\,{}^ts}{d\,{}^0s}\right)^2 \tag{11.7}$$

The length of a fiber is

$$d\,{}^0s = \left(d\,{}^0\boldsymbol{x}^T d\,{}^0\boldsymbol{x}\right)^{\frac{1}{2}} \tag{11.8}$$

$$\left({}^t\lambda\right)^2 = \frac{\left(d\,{}^0\boldsymbol{x}^T\,{}^t_0\boldsymbol{X}^T\right)\left({}^t_0\boldsymbol{X}\,d\,{}^0\boldsymbol{x}\right)}{d\,{}^0s\cdot d\,{}^0s}, \qquad \text{from (11.5)} \tag{11.9}$$

Express

$$d\,{}^0\boldsymbol{x} = \left(d\,{}^0s\right){}^0\boldsymbol{n} \tag{11.10}$$

$${}^0\boldsymbol{n} = \text{ unit vector into direction of } d\,{}^0\boldsymbol{x} \tag{11.11}$$

$$\Rightarrow \left({}^t\lambda\right)^2 = {}^0\boldsymbol{n}^T\,{}^t_0\boldsymbol{C}\,{}^0\boldsymbol{n} \tag{11.12}$$

$$\therefore \boxed{{}^t\lambda = \left({}^0\boldsymbol{n}^T\,{}^t_0\boldsymbol{C}\,{}^0\boldsymbol{n}\right)^{\frac{1}{2}}} \tag{11.13}$$

Also,

$$\left(d\,{}^t\hat{\boldsymbol{x}}\right)^T\cdot\left(d\,{}^t\boldsymbol{x}\right) = \left(d\,{}^t\hat{s}\right)\left(d\,{}^ts\right)\cos{}^t\theta, \qquad (\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos\theta) \tag{11.14}$$

From (11.5),

$$\cos{}^t\theta = \frac{\left(d\,{}^0\hat{\boldsymbol{x}}^T\,{}^t_0\hat{\boldsymbol{X}}^T\right)\left({}^t_0\boldsymbol{X}\;d\,{}^0\boldsymbol{x}\right)}{d\,{}^t\hat{s}\;d\,{}^ts} \qquad \left({}^t_0\hat{\boldsymbol{X}} \equiv {}^t_0\boldsymbol{X}\right) \tag{11.15}$$

$$= \frac{d\,{}^0\hat{s}\,{}^0\hat{\boldsymbol{n}}^T\,{}^t_0\boldsymbol{C}\,{}^0\boldsymbol{n}\,d\,{}^0s}{d\,{}^t\hat{s}\cdot d\,{}^ts} \tag{11.16}$$

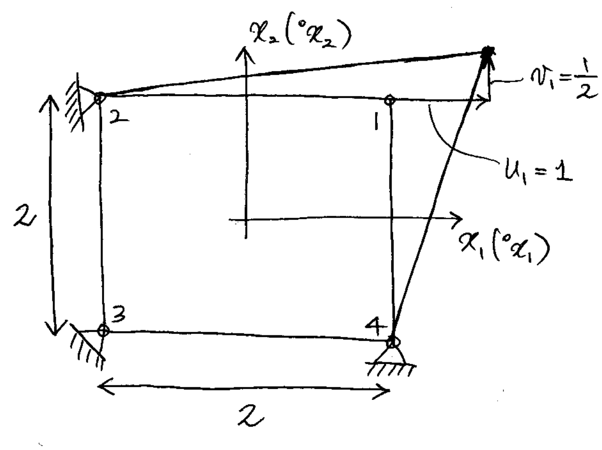
$$\therefore \boxed{\cos{}^t\theta = \frac{{}^0\hat{\boldsymbol{n}}^T\,{}^t_0\boldsymbol{C}\,{}^0\boldsymbol{n}}{{}^t\hat{\lambda}\,{}^t\lambda}} \tag{11.17}$$

Also,

$$\boxed{{}^t\rho = \frac{{}^0\rho}{\det{}^t_0\boldsymbol{X}}} \qquad \text{(see Ex. 6.5)} \tag{11.18}$$

**Example**

Reading: Ex. 6.6 in the text




$$h_1 = \frac{1}{4}(1 + {}^0x_1)(1 + {}^0x_2) \tag{11.19}$$

$$\vdots$$

$$ {}^t x_i = {}^0x_i + {}^t u_i \tag{11.20}$$

$$= \sum_{k=1}^{4} h_k \, {}^t x_i^k, \quad (i = 1, 2) \tag{11.21}$$

where ${}^t x_i^k$ are the nodal point coordinates at time $t$ (${}^t x_1^1 = 2$, ${}^t x_2^1 = 1.5$)

Then we obtain

$$ {}^t_0\boldsymbol{X} = \frac{1}{4} \begin{bmatrix} 5 + {}^0x_2 & 1 + {}^0x_1 \\ \frac{1}{2}(1 + {}^0x_2) & \frac{1}{2}(9 + {}^0x_1) \end{bmatrix} \tag{11.22}$$

At ${}^0x_1 = 0$, ${}^0x_2 = 0$,

$$ {}^t_0\boldsymbol{X}\Big|_{{}^0x_i = {}^0x_2 = 0} = \frac{1}{4} \begin{bmatrix} 5 & 1 \\ \frac{1}{2} & \frac{9}{2} \end{bmatrix} \tag{11.23}$$

**The Green-Lagrange Strain**

$$ {}^t_0\epsilon = \frac{1}{2}\left({}^t_0\boldsymbol{X}^T \, {}^t_0\boldsymbol{X} - \boldsymbol{I}\right) = \frac{1}{2}\left({}^t_0\boldsymbol{C} - \boldsymbol{I}\right) \tag{11.24}$$

$$\frac{\partial^t x_i}{\partial^0 x_j} = \frac{\partial\left({}^0x_i + {}^t u_i\right)}{\partial^0 x_j} = \delta_{ij} + \frac{\partial^t u_i}{\partial^0 x_j} \tag{11.25}$$

We find that

$$ {}^t_0\epsilon_{ij} = \frac{1}{2}\left({}^t_0u_{i,j} + {}^t_0u_{j,i} + {}^t_0u_{k,i}\,{}^t_0u_{k,j}\right), \qquad \text{sum over } k = 1, 2, 3 \tag{11.26}$$

where

$$ {}^t_0u_{i,j} = \frac{\partial^t u_i}{\partial^0 x_j} \tag{11.27}$$

**Polar decomposition of ${}_0^t\boldsymbol{X}$**

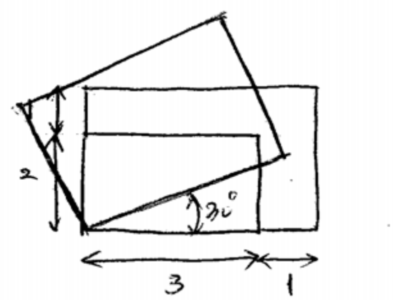
$$ {}_0^t\boldsymbol{X} = {}_0^t\boldsymbol{R}\,{}_0^t\boldsymbol{U} \tag{11.28} $$

where ${}_0^t\boldsymbol{R}$ is a rotation matrix, such that

$$ {}_0^t\boldsymbol{R}^T\,{}_0^t\boldsymbol{R} = \boldsymbol{I} \tag{11.29} $$

and ${}_0^t\boldsymbol{U}$ is a symmetric matrix (stretch)

**Ex. 6.9 textbook**




$$ {}_0^t\boldsymbol{X} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{4}{3} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \tag{11.30} $$

Then,

$$ {}_0^t\boldsymbol{C} = {}_0^t\boldsymbol{X}^T\,{}_0^t\boldsymbol{X} = \left({}_0^t\boldsymbol{U}\right)^2 \tag{11.31} $$

$$ {}_0^t\boldsymbol{\epsilon} = \frac{1}{2}\left[\left({}_0^t\boldsymbol{U}\right)^2 - \boldsymbol{I}\right] \tag{11.32} $$

This shows, by an example, that the components of the Green-Lagrange strain are independent of a rigid-body rotation.

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