Part 2

Massachusetts Institute of Technology
Department of Mechanical Engineering

2.12 Introduction to Robotics
Exercise problems and Solutions for the End-of-Term Exam

Problem 1

Figure 1 shows a two d.o.f. planar robot with a parallelogram mechanism. Note that links 1 and 3 as well as 2 and 4 are parallel to each other, and that joint axes 1 and 2 are aligned. Links 1 and 2 are driven by independent DC motors fixed to the base. Using the notations shown in the figure, answer the following questions.

[1-1] Obtain the Jacobian matrix relating endpoint velocities v_x and v_y to joint velocities $\dot{\theta}_1$ and $\dot{\theta}_2$.

[1-2] Obtain the joint torques, τ_1 and τ_2, required for bearing endpoint forces, F_x and F_y.

[1-3] Let K_p be a $2x2$ feedback gain matrix relating joint torques $\tau = [\tau_1, \ \tau_2]^T$ to joint position error $\delta\theta = [\delta\theta_1, \ \delta\theta_2]^T$, that is, $\tau = K_p \cdot \delta\theta$. Obtain the gain matrix K_p that generates the desired endpoint compliance specified in Figure 2. Namely, the desire compliance is C_1 and C_2 in the two directions of principal axis, x_b and y_b, while the axis x_b is angle α from the x axis.

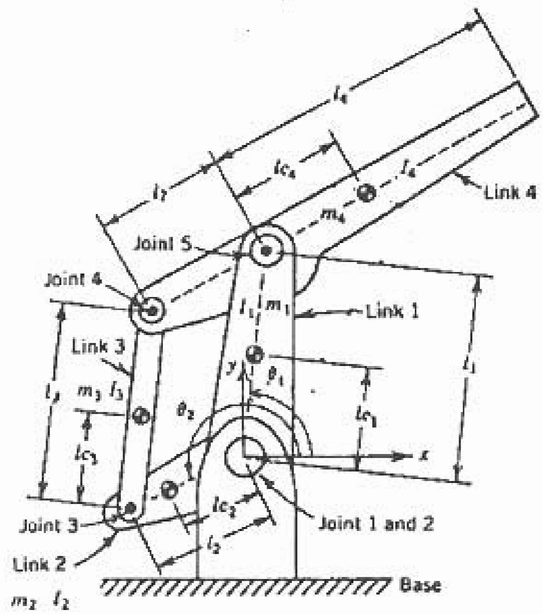


Figure 1 Planar robot with parallelogram mechanism

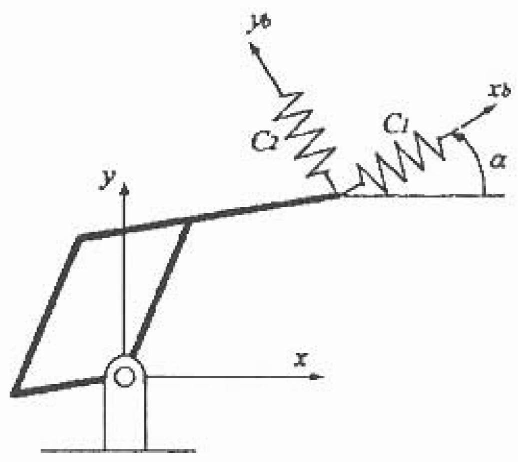


Figure 2 Desired compliance

[1-4]. Obtain the moment of inertia reflected to Joint 1 when Joint 2 is fixed, that is, the 1-1 element of the inertia matrix $\mathbf{H} = \{H_{ij}\}$ associated with generalized coordinates θ_1, θ_2. Which particular motion of the arm links is the H_{11} element associated with?

[1-5]. Explain why H_{11} and H_{22} of the inertia matrix do not vary, although the arm configuration varies, i.e. configuration-invariant.

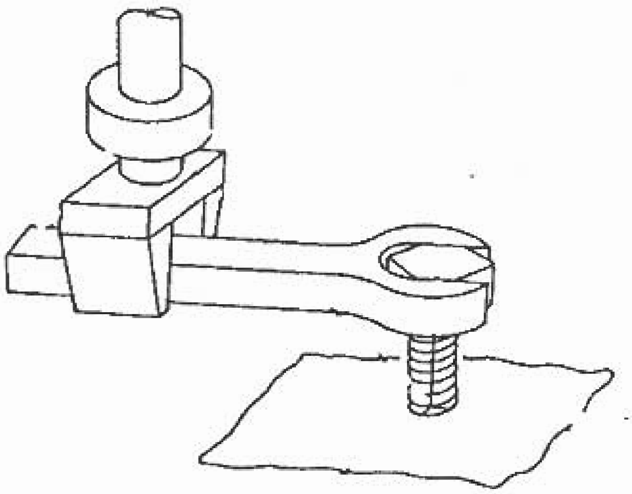
[1-6]. Show that the off-diagonal element of the inertia matrix is given by

$$H_{12} = (m_3 \ell_2 \ell_{C3} - m_4 \ell_1 \ell_{C4}) \cos(\theta_1 - \theta_2).$$

[1-7]. We want to decouple and linearize the arm dynamics by re-distributing mass among the four links. Obtain conditions for the mass parameters to make the arm dynamics totally decoupled and linearized for all arm configurations. Also explain why that happens.

Problem 2

A robot is unscrewing a hexa-head bolt by using a wrench, as shown in the figure below. Obtain both natural and artificial constraints to perform this task by using Mason's hybrid position/force control. Define an appropriate C-frame and parameters necessary for describing the constraints, and show your notations in a sketch. Assume that the process is quasi-static and no friction. Also assume that there is no gap between the head of the bolt and the wrench.



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Solutions

Problem 1

(1-1) Endpoint coordinates are given by

$$\begin{cases} x = \ell_1 \cos\theta_1 + \ell_4 \cos(\theta_2 - \pi) \\ y = \ell_1 \sin\theta_1 + \ell_4 \sin(\theta_2 - \pi) \end{cases}$$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = \underbrace{\begin{bmatrix} -\ell_1 \sin\theta_1, & \ell_4 \sin\theta_2 \\ \ell_1 \cos\theta_1, & -\ell_4 \cos\theta_2 \end{bmatrix}}_{J} \begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \end{pmatrix}$$

(1-2). From duality

$$\tau = J^T F.$$

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{bmatrix} -\ell_1 \sin\theta_1 & \ell_1 \cos\theta_1 \\ \ell_4 \sin\theta_2 & -\ell_4 \cos\theta_2 \end{bmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

(1-3) Let Δp and Δp^b be endpoint displacements in the x-y and x^b-y^b coordinate systems, respectively, and F and F^b the corresponding endpoint forces. The desired endpoint stiffness in the x^b-y^b system is given by

$$F^b = (C_d^b)^{-1} \Delta p^b$$

where

$$C_d^b = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$$ is the desired compliance matrix.

This must be represented in the x-y coordinate system by using the coordinate transformation given by

$$\Delta p = R \Delta p^b \quad \text{or} \quad \Delta p^b = R^T \Delta p$$

where

$$R = \begin{pmatrix} \cos\alpha, & -\sin\alpha \\ \sin\alpha; & \cos\alpha \end{pmatrix}$$

From duality

$$F = R F^b.$$

Combining all these yields

$$F = R (C^b_d)^{-1} R^T \Delta p, \qquad R(C^b_d)^{-1} R^T = \text{Desired stiffness in } x\text{-}y.$$

Let J be the Jacobian matrix relating Δp to $\Delta\theta$ obtained in (1-1)

$$\tau = J^T F = J^T R (C^b_d)^{-1} R^T J \Delta\theta.$$

The matrix products yield the joint feedback gain matrix:

$$K_p = J^T R (C^b_d)^{-1} R^T J$$

(1-4) and (1-5)

Physical Sense : H_{11} represents the inertia seen by the first actuator (joint 1) when joint 2 is immobilized; $\dot{\theta}_2 = 0, \ddot{\theta}_2 = 0$.

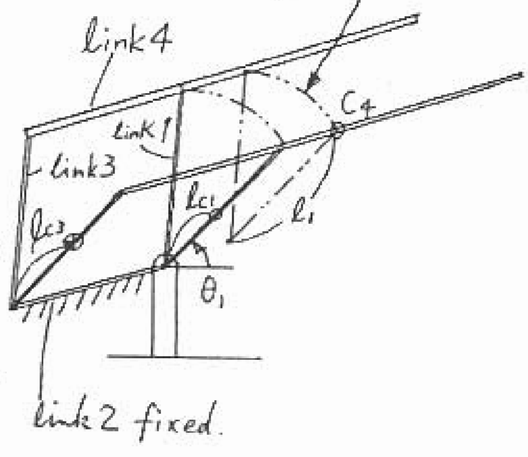
Link movements associated with H_{11}

Link 1 rotates about joint 1, hence the effective inertia at joint 1 is $I_1 + m_1 l_{c1}^2$, which is constant.

Link 2 stationary.

Link 3 rotates about joint 3, which is stationary when joint 2 is immobilized. The effective inertia is therefore $I_3 + m_3 l_{c3}^2$, which is configuration independent.

Link 4 is kept parallel to link 2, which is stationary. Therefore link 4 does not rotate but translate along the circular trajectory shown below. The radius of this trajectory is l_1 for an arbitrary arm configuration. Therefore the effective inertia is $m_4 l_1^2$, which does not depend on θ_1, θ_2. In all these no joint variables are involved, hence H_{11} is configuration-invariant.



(1-6). H_{12} is the coupling inertia between Joints 1 and 2.

consider the instant when $\dot{\theta}_1 = 0$, $\dot{\theta}_2 = 0$ and $\ddot{\theta}_2 = 0$ and Joint 1 alone is accelerated $\ddot{\theta}_1 \neq 0$.

Due to symmetry, we should get the same result when we set $\ddot{\theta}_2 \neq 0$, and $\dot{\theta}_1 = \ddot{\theta}_1 = 0$;

The H_{12} term represents the reaction torque at joint 1 when link 1 is immobilized.

From (1-5) we know that link 4 only undergoes translation. The inertia due to links 3 and 1 causes torques $I_3\ddot{\theta}_1$ and $I_1\ddot{\theta}_1$, both equal to zero. Thus, the only torque contribution will be due to body forces of links 3 + 4.

$$\tau_{int} = \mathbf{r}_{0,3} \times \mathbf{f}_{3,2} + \mathbf{r}_{0,4} \times \mathbf{f}_{4,1}$$

$$= \mathbf{r}_{0,c_3} \times -m_3\dot{\mathbf{v}}_{c_3} + \mathbf{r}_{0,c_4} \times -m_4\dot{\mathbf{v}}_{c_4}$$

For link 3,

$$\mathbf{r}_{0,c_3} = \begin{pmatrix} l_2\cos\theta_2 + l_{c_3}\cos\theta_1 \\ l_2\sin\theta_2 + l_{c_3}\sin\theta_1 \end{pmatrix}$$

$$\dot{\mathbf{v}}_{c_3}\Big|_{\ddot{q}_1 = \dot{q}_1 = 0} = \begin{pmatrix} -l_2\sin\theta_2\ddot{\theta}_2 \\ l_2\cos\theta_2\ddot{\theta}_2 \end{pmatrix}$$

Similarly for link 4,

$$\mathbf{r}_{0,c4} = \begin{pmatrix} l_1 \cos\theta_1 + l_{c4}\cos(\theta_2 - \pi) \\ l_1 \sin\theta_1 + l_{c4}\sin(\theta_2 - \pi) \end{pmatrix}$$

$$\dot{\mathbf{v}}_{c4}\Big|_{\dot{q}=\dot{q}=0} = \begin{pmatrix} l_{c4}\sin\theta_2\ddot{\theta}_2 \\ -l_{c4}\cos\theta_2\ddot{\theta}_2 \end{pmatrix}$$

Now, we can write,

$$\tau_{1nt} = m_3 \begin{vmatrix} l_2\cos\theta_2 + l_{c3}\cos\theta_1 , & -l_2\sin\theta_2\ddot{\theta}_2 \\ l_2\sin\theta_2 + l_{c3}\sin\theta_1 , & l_2\cos\theta_2\ddot{\theta}_2 \end{vmatrix}$$
$$+ m_4 \begin{vmatrix} l_1\cos\theta_1 + l_{c4}\cos(\theta_2 - \pi) , & l_{c4}\sin\theta_2\ddot{\theta}_2 \\ l_1\sin\theta_1 + l_{c4}\sin(\theta_2 - \pi) , & -l_{c4}\cos\theta_2\ddot{\theta}_2 \end{vmatrix}$$

$$\tau_{1nt} = m_3\left\{ l_2 l_{c3}(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2)\right\}\ddot{\theta}_2$$
$$+ m_4\left\{ l_1 l_{c4}(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2)\right\}\ddot{\theta}_2$$

$$\tau_{nt} = m_3 l_2 l_{c3}\cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_4 l_1 l_{c4}\cos(\theta_1 - \theta_2)\ddot{\theta}_2$$

$$\Rightarrow \quad H_{12} = (m_3 l_2 l_{c3} + m_4 l_1 l_{c4})\cos(\theta_1 - \theta_2)$$

(1-7) From (1-6), the off-diagonal element of the inertia matrix is

$$H_{12} = (m_3 l_2 l_{c3} - m_4 l_1 l_{c4}) \cos(\theta_1 - \theta_2)$$

This vanishes when

i) $m_3 l_2 l_{c3} - m_4 l_1 l_{c4} = 0$. or.

ii) $\cos(\theta_1 - \theta_2) = 0$ i.e. $\theta_1 - \theta_2 = \pm \pi/2$.

When only ii) is held, H_{12} becomes zero instantaneously only when $\theta_1 - \theta_2 = \pm \pi/2$. Therefore the time derivative of H_{12} is not necessarily zero even at $\theta_1 - \theta_2 = \pm\pi/2$. Namely, the arm dynamics is not linear although the coupling becomes zero when $\theta_1 - \theta_2 = \pm \pi/2$.

On the other hand when condition i) is held, namely,

$$\boxed{\frac{m_3}{m_4} = \frac{l_1}{l_2} \cdot \frac{l_{c4}}{l_{c3}}},$$

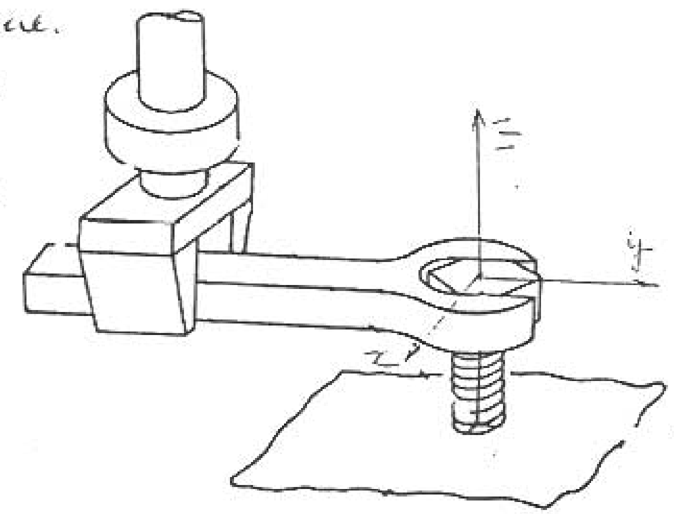
The inertia matrix reduces to a constant, diagonal matrix :

$$H^* = \begin{bmatrix} I_1 + I_3 + m_1 l_{c1}^2 + m_3 l_{c3}^2\left(1 + \frac{l_1 l_2}{l_{c3} l_{c4}}\right), & 0 \\ 0 & , I_2 + I_4 + m_2 l_{c2}^2 + m_3 l_2^2\left(1 + \frac{l_{c3} l_{c4}}{l_1 l_2}\right) \end{bmatrix}$$

Since H^* is constant for all arm configurations, there is no nonlinear force other than gravity terms. Therefore if the above mass distribution condition is held, the system becomes totally decoupled and linear for all arm configuration.

Problem 2

Use the C frame shown in the figure.
Note that the wrench can slide in the z direction relative to the bolt head. As the bolt is rotated by the wrench, the bolt moves in the z direction; $p\Omega$ = (pitch) × (angle of rotation).
It is desirable (not demanded) to move the wrench accordingly, otherwise the wrench slides off the bolt head.



	Kinematic	Static
Natural	$v_x = 0$ $\omega_x = 0$ $v_y = 0$ $\omega_y = 0$.	$\tau_z = 0$ $f_z = 0$.
Artificial	$\omega_z = \Omega > 0$ $v_z = p\Omega > 0$.	$f_x = 0$ $\tau_x = 0$ $f_y = F \geqq 0$. $\tau_y = 0$.

Anothe solution which is acceptable:

	Kinematic	Static
Natural	$v_x = 0$ $\omega_y = 0$.	$f_y = 0$ $\tau_x = 0$ $f_z = 0$. $\tau_z = 0$.
Artificial	$v_y = 0$ $\omega_x = 0$ $v_z = p\Omega$ $\omega_z - \Omega$	$f_x = 0$ $\tau_y = 0$

– In this case the wrench is not fully inserted to the bolt head, but has a play in the y direction.