

## 2.141 Modeling and Simulation of Dynamic Systems

### Assignment #4

Out: Wednesday October 25, 2006

Due: Monday November 6, 2006

#### Problem 1

A bead of mass  $m$  is attached to the center of a wire of length  $2L$  which is rigidly restrained at its ends. The wire has longitudinal elastic stiffness  $k$  and pre-tension  $T_0$  (the tension in the wire when it is perfectly straight).

1. Develop a model competent to describe horizontal-plane transverse vibration of the mass and represent it as a bond graph.
2. Derive (nonlinear) equations of motion.
3. Linearize them about the rest configuration (when the wire is perfectly straight) and find an algebraic expression for the undamped frequency of oscillation.
4. How much does the elastic stiffness of the wire affect the frequency?

#### Problem 2

*Motivation: Control through Singularities*

A common form of robot motion control specifies a workspace position or trajectory (e.g., a desired time-course of tool position in Cartesian coordinates) and transforms that specification to a corresponding configuration-space position or trajectory (e.g., a time-course of joint angles). However, most robot mechanisms have kinematic singularities, configurations at which the relation between workspace and configuration-space becomes ill-defined. As a result, most robot motion controllers do not operate at or near these singular points. In contrast, humans frequently operate their limbs at or near “singular” configurations. (For example, think about your leg posture when you are standing).

An energy-based analysis of mechanics shows that the transformation of positions and velocities is well-defined in one direction while the transformation of efforts and momenta is well-defined in the other. A controller that takes advantage of these facts should be able to operate at and close to mechanism singularities. A “simple” impedance controller attempts to impose the workspace behavior of a damped spring connected to a movable “virtual position”.

$$\boldsymbol{\tau} = \mathbf{J}'(\boldsymbol{\theta})\{\mathbf{K}(\mathbf{x}_o - \mathbf{L}(\boldsymbol{\theta})) + \mathbf{B}(\dot{\mathbf{x}}_o - \mathbf{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}})\}$$

where  $\boldsymbol{\theta}$  is a vector of generalized coordinates,  $\boldsymbol{\tau}$  a vector of (conjugate) generalized efforts,  $\mathbf{x}$  and  $\mathbf{x}_o$  are vectors of actual and virtual Cartesian tip coordinates,  $\mathbf{L}(\cdot)$  and  $\mathbf{J}(\cdot)$  are linkage kinematic equations and Jacobian respectively, and  $\mathbf{K}$  and  $\mathbf{B}$  are stiffness and damping respectively.

Note that this controller does not require the inverse of the kinematic equations nor the Jacobian. In this problem you are to test how well works near and close to mechanism singularities.

Assume a planar serial-link (open-chain) mechanism with two links of equal length  $L = 0.5$  m, operating in the horizontal plane (i.e., ignore gravity) and driven by

ideal controllable-torque actuators, one driving the inner (“shoulder”) link relative to ground, the other driving the outer (“elbow”) link relative to the inner link. Sensors mounted co-axially with the actuators provide measurements of joint angle and angular velocity.

## 1. Kinematics

- Choose generalized coordinates (carefully!) and write kinematic equations relating them to the position coordinates of the tip, expressed in a Cartesian coordinate frame with its origin at the axis connecting the inner (“shoulder”) link to ground.
- Find the corresponding Jacobian (to relate generalized velocities to tip Cartesian velocities).
- Identify the set of singular configurations for this linkage (at which the relation between tip Cartesian coordinates and joint angles becomes ill-defined) and show that they include the center of the workspace as well as at the limits of reach.

## 2. Dynamics

- Formulate a *dynamic* model of the mechanism relating input actuator torques to output motion of the tip in Cartesian coordinates. Assume the links are rods of uniform cross section and mass  $m = 0.5$  kg and that the joints are frictionless. (**Hint:** be careful in your choice of generalized coordinates and be sure you have correctly identified generalized forces.)

## 3. Controller

- Assume a simple impedance controller with uniform tip stiffness and damping (i.e., the stiffness and damping matrices have the form:  $\mathbf{K} = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  and

$$\mathbf{B} = b \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ where } k \text{ and } b \text{ are constants). Choose the stiffness and damping}$$

values so that when the mechanism is making small motions about a configuration with the inner link aligned along the Cartesian  $x$ -axis and the outer link aligned parallel to the Cartesian  $y$ -axis, the highest-bandwidth transfer function between virtual and actual position has critically-damped poles with an undamped natural frequency of 2 Hz. (**Hint:** it may be easiest to first transform the stiffness and damping to generalized coordinates.)

## 4. Simulation

Simulate the response to the following virtual trajectories and plot at least the path of the tip in Cartesian coordinates:

- Starting from rest at  $\{x = L, y = 0\}$  ending at rest at  $\{x = 2L, y = 0\}$ ; trapezoidal speed profile with acceleration to peak speed in 250 ms, constant speed for 1.5 sec and deceleration to rest in 250 ms.
- Starting from rest at  $\{x = -L, y = 0\}$  ending at rest at  $\{x = L, y = 0\}$ ; trapezoidal speed profile with acceleration to peak speed in 250 ms, constant speed for 1.5 sec and deceleration to rest in 250 ms.

- c) Starting from rest at  $\{x = -L, y = L/20\}$  ending at rest at  $\{x = L, y = L/10\}$ ; trapezoidal speed profile with acceleration to peak speed in 250 ms, constant speed for 1.5 sec and deceleration to rest in 250 ms.
- d) Repeat the last simulation ( $\{x = -L, y = L/20\}$  to  $\{x = L, y = L/10\}$ ) with  $k$  and  $b$  chosen so that the highest-bandwidth transfer function between virtual and actual position has critically-damped poles with an undamped natural frequency of 20 Hz..

Comment briefly on whether and how well this controller operates near mechanism singularities.