HAMILTON-JACOBI THEORY

GOAL:

Find a particular canonical transformation such that the "new" Hamiltonian is a function only of the "new" momenta.

MATHEMATICAL PRELIMINARIES

A canonical transformation may be derived from a *generating function*.

Arguments of a generating function mix "old" and "new" variables.

e.g., old momenta, new displacements

S(q*,p)

Differentiation yields "old" displacements and "new" momenta.

$p^* = -\partial S/\partial q^*$

$q = -\partial S/\partial p$

There are three other possible generating functions:

S(p*,q)

$q^* = \partial S/\partial p^*$

$p = \partial S/\partial q$

S(q*,q)

$p^* = -\partial S/\partial q^*$

$p = \partial S/\partial q$

S(p*,p)

$q^* = \partial S/\partial p^*$

$q = -\partial S/\partial p$

"OLD" HAMILTONIAN

$H(q_1, \ldots, q_n, p_1, \ldots, p_n)$

To find the required transformation to the "new" variables, use a generating function

$S(q_1, \ldots, q_n, p^*_1, \ldots, p^*_n)$

from which

$p_j = \partial S/\partial q_j$

"NEW" HAMILTONIAN

Substitute into the "old" Hamiltonian

$$K(p^*_1, \ldots, p^*_n) = H(q_1, \ldots, q_n, \partial S/\partial q_1, \ldots, \partial S/\partial q_n)$$

This is a partial differential equation defining $S(\cdot)$ as a function of $\mathbf{q} = [q_1, \ldots, q_n]^t$.

For the purpose of solving this equation, the "new" momenta $\mathbf{p}^* = [p^*_1, \ldots, p^*_n]^t$ and the "new" Hamiltonian $K(\mathbf{p}^*)$ may be treated as constant parameters.

$$H(q_1, \ldots, q_n, \partial S/\partial q_1, \ldots, \partial S/\partial q_n) = \text{constant}$$

This is a special case of the Hamilton-Jacobi equation.

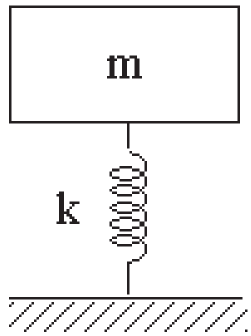
Its solution defines the required transformation.

Aside:

The Hamilton-Jacobi equation plays a prominent role in optimal control theory.

EXAMPLE:

A simple harmonic oscillator



$$H(q,p) = \left(\frac{p^2}{2m} + \frac{kq^2}{2}\right) = \frac{1}{2m}(p^2 + mkq^2) = \frac{1}{2m}(p^2 + Z^2q^2)$$

where $Z = \sqrt{km}$

set $p = \partial S/\partial q$ and substitute

$$H(q,\partial S/\partial q) = \frac{1}{2m}((\partial S/\partial q)^2 + Z^2q^2) = \text{constant} = K(p^*)$$

$$\partial S/\partial q = (2mK(p^*) - Z^2q^2)^{1/2}$$

— a partial differential equation for $S(q)$

Choose $K(p^*) = \omega p^*$

where $\omega = \sqrt{k/m}$

$$\partial S/\partial q = (2Zp^* - Z^2q^2)^{1/2}$$

$$S = \int(2Zp^* - Z^2q^2)^{1/2}dq$$

differentiate to find q*

$$q^* = \partial S/\partial p^* = \int \frac{Z dq}{\sqrt{2Zp^* - Z^2 q^2}}$$

substitute $u = \frac{q}{\sqrt{2p^*/Z}}$

$$q^* = \int \frac{du}{1-u^2} = \sin^{-1}(u) = \sin^{-1}\left(\frac{q}{\sqrt{2p^*/Z}}\right)$$

$$q = \sqrt{2p^*/Z}\ \sin(q^*)$$

$$p = \partial S/\partial q = (2Zp^* - Z^2 q^2)^{1/2} = \sqrt{2Zp^*(1 - \sin^2(q^*))}$$

$$p = \sqrt{2Zp^*}\ \cos(q^*)$$

New equations

$$dp^*/dt = -\,\partial K(p^*)/\partial q^* = 0$$

$$dq^*/dt = \partial K(p^*)/\partial p^* = \omega$$

thus

$$p^* = \text{constant}$$

$$q^* = \omega t + \text{constant}$$

The transformation $q = \sqrt{2p^*/Z}\ \sin(q^*)$ and $p = \sqrt{2Zp^*}$ $\cos(q^*)$ *integrates the differential equations*