INERTIAL MECHANICS

Neville Hogan

The inertial behavior of a mechanism is substantially more complicated than that of a translating rigid body. Strictly speaking, the dynamics are simple; the underlying mechanical physics is still described by Newton's laws. The complexity arises from the kinematic constraints between the motions of its members. One powerful method to describe inertial mechanics is Lagrange's equation, which is traditionally introduced using the variational calculus with Hamilton's principle of stationary action. Here's a more direct approach that may provide more insight.

LAGRANGE'S EQUATION FOR MECHANISMS

Begin with the uncoupled members of the mechanism.

- **x** uncoupled coordinates (orientations, locations of mass centers) with respect to a non-accelerating (inertial) reference frame
- **v** velocities
- **p** momenta
- **f** forces

These four fundamental quantities are related as follows.

$$d\mathbf{x}/dt = \mathbf{v}$$

$$d\mathbf{p}/dt = \mathbf{f}$$

The constitutive equation for kinetic energy storage (inertia) is:

$$\mathbf{p} = \mathbf{M}\mathbf{v}$$

M *diagonal* matrix of inertial parameters (masses, moments of inertia, e.g. about mass centers)

Kinetic *co*-energy is the dual of kinetic energy:

$$E_k^* = \int \mathbf{p}^t d\mathbf{v} = \frac{1}{2}\ \mathbf{v}^t\, \mathbf{M}\mathbf{v} = E_k^*(\mathbf{v})$$

Thus, by definition:

$$\mathbf{p} = \partial E_k^* / \partial \mathbf{v}$$

Aside:

The underlying mechanical physics is fundamentally independent of choice of coordinates. Therefore, these may be regarded as *tensor* equations. By the usual conventions:

v is a contravariant rank 1 tensor (vector)

M is a twice co-variant rank 2 tensor

p is covariant rank 1 tensor (vector)

These observations become more useful when we consider transformations of variables.

Next consider the kinematically coupled mechanism.

θ generalized coordinates (or configuration variables)

— a (non-unique) set of independent variables that uniquely and completely define the (mechanism) configuration

ω generalized velocities

— the time derivatives of generalized coordinates.

$d\theta/dt = \omega$

τ generalized forces (moments or torques)

η generalized momenta

The relation between generalized forces and momenta requires care. If the kinematic constraints are *holonomic*, the relation between coordinates is a set of algebraic equations[1].

$$\mathbf{x} = \mathbf{L}(\theta)$$

Relation between velocities:

$$d\mathbf{x}/dt = \left(\partial \mathbf{L}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}\right) d\theta/dt$$

$$\mathbf{v} = \mathbf{J}(\theta)\boldsymbol{\omega}$$

where $\mathbf{J}(\theta) = \left(\partial \mathbf{L}(\boldsymbol{\theta}) / \partial \boldsymbol{\theta}\right)$

[1] *Non-holonomic* constraints are commonplace. A typical example is a constraint between velocities that cannot be integrated to a constraint between coordinates.

The relation between generalized forces may be derived from power continuity[2] (a differential statement of energy conservation).

Power:

$$P = \boldsymbol{\tau}^t \boldsymbol{\omega}$$

Power continuity:

$$P = \boldsymbol{\tau}^t \boldsymbol{\omega} = \mathbf{f}^t \mathbf{v} = \mathbf{f}^t \mathbf{J}(\theta)\boldsymbol{\omega}$$

This must be true for all values of $\boldsymbol{\omega}$, therefore

$$\boldsymbol{\tau} = \mathbf{J}(\theta)^t \mathbf{f}$$

Aside:

A common error is to mis-identify generalized forces. The relation between power, generalized force and generalized velocity is a rigorous and reliable definition of generalized forces.

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[2] This avoids the sometimes-confusing *principle of virtual work* but is completely equivalent.

The relation between kinetic co-energies may be obtained by substitution using the relation between velocities.

$$E_k^* = \frac{1}{2}\ \omega^t\, \mathbf{J}(\theta)^t\, \mathbf{MJ}(\theta)\omega$$

Kinetic energy in generalized coordinates is a quadratic form in velocity. The kernel of the quadratic form is the *inertia tensor*.

$$\mathbf{I}(\theta) = \mathbf{J}(\theta)^t\, \mathbf{MJ}(\theta)$$

$$E_k^* = \frac{1}{2}\ \omega^t\, \mathbf{I}(\theta)\omega$$

Note that kinetic co-energy, which previously was a function of velocity alone, is now a function of velocity and position.

$$E_k^* = E_k^*(\theta,\omega)$$

This is the main reason why (to paraphrase Prof. Stephen Crandall) “mechanics is hard for humans”.

The relation between momenta follows directly. Generalized momenta are defined as before.

$$\boldsymbol{\eta} = \partial E_k^{*} / \partial \boldsymbol{\omega}$$

$$\boldsymbol{\eta} = \mathbf{I}(\theta)\omega = \mathbf{J}(\theta)^t\, \mathbf{M}\mathbf{J}(\theta)\omega$$

$$\boldsymbol{\eta} = \mathbf{J}(\theta)^t\, \mathbf{M}\mathbf{v} = \mathbf{J}(\theta)^t\, \mathbf{p}$$

$$\boldsymbol{\eta} = \mathbf{J}(\theta)^t\, \mathbf{p}$$

KEY POINT:

Generalized force is *not* the derivative of generalized momentum

$$d\boldsymbol{\eta}/dt \neq \boldsymbol{\tau}$$

Differentiate the relation between momenta

$$d\boldsymbol{\eta}/dt = \mathbf{J}(\boldsymbol{\theta})^t\, d\mathbf{p}/dt + \boldsymbol{\omega}^t\, [\partial\mathbf{J}(\boldsymbol{\theta})^t /\partial\boldsymbol{\theta}]\mathbf{p}$$

$$d\boldsymbol{\eta}/dt = \mathbf{J}(\boldsymbol{\theta})^t\, \mathbf{f} + \boldsymbol{\omega}^t\, [\partial\mathbf{J}(\boldsymbol{\theta})^t /\partial\boldsymbol{\theta}]\mathbf{M}\mathbf{J}(\boldsymbol{\theta})\boldsymbol{\omega}$$

The second term appears to be related to the kinetic co-energy. It is:

$$\partial E_k^*/\partial\boldsymbol{\theta} = \frac{1}{2}\, \boldsymbol{\omega}^t\, [\partial\mathbf{J}(\boldsymbol{\theta})^t /\partial\boldsymbol{\theta}]\mathbf{M}\mathbf{J}(\boldsymbol{\theta})\boldsymbol{\omega} + \frac{1}{2}\, \boldsymbol{\omega}^t\, \mathbf{J}(\boldsymbol{\theta})^t\, \mathbf{M}[\partial\mathbf{J}(\boldsymbol{\theta})/\partial\boldsymbol{\theta}]\boldsymbol{\omega}$$

$$\partial E_k^*/\partial\boldsymbol{\theta} = \boldsymbol{\omega}^t\, [\partial\mathbf{J}(\boldsymbol{\theta})^t /\partial\boldsymbol{\theta}]\mathbf{M}\mathbf{J}(\boldsymbol{\theta})\boldsymbol{\omega}$$

$$d\boldsymbol{\eta}/dt = \boldsymbol{\tau} + \partial E_k^*/\partial\boldsymbol{\theta}$$

This is *Lagrange's equation*

$$d\boldsymbol{\eta}/dt - \partial E_k^*/\partial\boldsymbol{\theta} = \boldsymbol{\tau}$$

It may be more familiar in expanded form. Identify kinetic co-energy with the Lagrangian, $L(\theta,\omega)$

$$L(\theta,\omega) = E_k^*(\theta,\omega)$$

$$\frac{d}{dt}\left[\frac{\partial E_k^*}{\partial \boldsymbol{\omega}}\right] - \frac{\partial E_k^*}{\partial \boldsymbol{\theta}} = \boldsymbol{\tau}$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \boldsymbol{\omega}}\right] - \frac{\partial L}{\partial \boldsymbol{\theta}} = \boldsymbol{\tau}$$

A SCALAR EXAMPLE:

$$x = L(\theta)$$

$$v = J(\theta)\omega$$

$$\tau = J(\theta)f$$

$$E_k^* = \frac{1}{2}\ mv^2 = \frac{1}{2}\ mJ(\theta)^2\omega^2$$

$$\eta = \partial E_k^*/\partial\omega = mJ(\theta)^2\omega = J(\theta)mv = J(\theta)p$$

$$d\eta/dt = J(\theta)dp/dt + \left[\partial J(\theta)/\partial\theta\right]\omega p$$

$$d\eta/dt = \tau + \left[\partial J(\theta)/\partial\theta\right]\omega mJ(\theta)\omega$$

$$\partial E_k^*/\partial\theta = mJ(\theta)\omega^2\partial J(\theta)/\partial\theta$$

$$d\eta/dt = \tau + \partial E_k^*/\partial\theta$$

SUMMARIZING:

	inertial coordinates	relation	generalized coordinates
displacement	$\mathbf{x}$	$\mathbf{x} = \mathbf{L}(\theta)$	θ
flow	$\mathbf{v}$	$\mathbf{v} = \mathbf{J}(\theta)\boldsymbol{\omega}$	$\boldsymbol{\omega}$
effort	$\mathbf{f}$	$\boldsymbol{\tau} = \mathbf{J}(\theta)^t\, \mathbf{f}$	$\boldsymbol{\tau}$
momentum	$\mathbf{p}$	$\eta = \mathbf{J}(\theta)^t\, \mathbf{p}$	η
inertia tensor	$\mathbf{M}$	$\mathbf{I}(\theta) = \mathbf{J}(\theta)^t\, \mathbf{M}\mathbf{J}(\theta)$	$\mathbf{I}(\theta)$
constitutive equation	$\mathbf{p} = \mathbf{M}\mathbf{v}$		$\eta = \mathbf{I}(\theta)\boldsymbol{\omega}$
kinetic co-energy	$\frac{1}{2}\ \mathbf{v}^t\, \mathbf{M}\mathbf{v}$		$\frac{1}{2}\ \boldsymbol{\omega}^t\, \mathbf{I}(\theta)\boldsymbol{\omega}$
	$d\mathbf{x}/dt = \mathbf{v}$		$d\theta/dt = \boldsymbol{\omega}$
	$d\mathbf{p}/dt = \mathbf{f}$		$d\eta/dt = \boldsymbol{\tau} + \partial E_k{}^*/\partial\theta$

The key ideas behind the Lagrangian formulation:

1. Incorporate holonomic kinematic constraints directly.
2. Write the momentum balance equation in terms of a state function, the kinetic co-energy.

Advantages:

1. Velocities are easily identified and kinetic co-energy is easily computed.
2. There is no need to write explicit expressions for the forces of constraint.

Disadvantages:

1. The kinetic co-energy is a quadratic form whose kernel typically contains trigonometric functions of sums of coordinates. Differentiating trigonometric functions of sums of coordinates breeds terms very rapidly. The Lagrangian approach requires a partial derivative of the co-energy followed by a total derivative of the co-energy. The algebraic complexity of the result can be staggering.
2. The Lagrangian approach is fundamentally a 2° form. To achieve the 1° form required for a state-determined representation (useful for numerical integration and most mathematical analysis) the inertia tensor must be inverted. This is anything but trivial.