Here's a set of excellent questions from Jooeun Ahn (and my answers).

"While working on homework problems of 2.141, I came to realize that I don't have rigorous knowledge about the definition of system order, model order and their relation with the number of state variables ...

... So, would you please teach me the exact definition of
1) system order
2) model order
and their relation with
3) the minimal number of state variables
4) the energy storage element that has integral causality

Also, I would like to learn
5) any systematic way to choose the proper state variables

Lastly, I believe that minimal number of generalized coordinate and degree of freedom are different from number of state variables or system order.
6) However, are they related to the order of system or number of state variables in any sense?
7) What is the exact definition of degree of freedom in system dynamics?

Jooeun"

I can't think of any reliable basis to distinguish between "system order" and "model order", partly because the term "system" is used so loosely. However, the following may help clarify matters.

Consider a model comprised of a single point mass translating in one dimension driven by a controllable force and retarded by some friction process related to its motion. A model of this "system" consists of the mass (a kinetic energy-storage element or inertance); the friction (a power dissipation element or resistance, possibly nonlinear); the force (an effort source); and the connection between them (common velocity, described by a one-junction).

The energy stored in this system is described by one variable, e.g., the momentum of the mass. The effort source may add or subtract energy but doesn't yield any interesting dynamics. By that I mean that if you specify any time-history of motion I can figure out the time-history of force required to produce it. There's no "memory" of prior states involved as in a dynamic system. The minimal dynamic behavior of this system arises from interaction between energy storage (due to motion) and power dissipation (due to motion). This behavior is evident even in the absence of forcing (i.e., force identically zero) and is described by a first-order ordinary differential equation, i.e., a state equation with one state variable.

You can deduce that by assigning causality to the bond-graph representation of the model. When you go to integrate differential equations, each independent energy-storage element will require one initial condition. The number of independent energy-storage elements is the minimal system (or model) order, one in this case. The state variable you choose is not unique but must be sufficient to determine the energy stored in the mass (e.g., its velocity or momentum).

However, you can easily find cases in which more differential equations are required. For example, if I want to know where the mass is (maybe I want to control its position) I need another differential equation. Thus if my state variable above was velocity, I need to integrate it to get position, and that integration operation requires one more initial condition, hence one more state variable. In this case I might argue that the "model" is the same as before (one mass, one dissipator and one source connected by a common-velocity junction) but the "system" includes the additional math needed to find position. However, I don't think the terminology of "system" or "model" is as precisely defined as this example might suggest.

It doesn't stop there. Suppose I wanted to design an integral-action controller to regulate the mass position and I decide to try state-space methods such as pole-placement. I need to define a state variable to keep track of the integrated error. Using a self-explanatory notation:

$$\dot{v} = \frac{1}{m}\left(F_{source} - F_{friction}\right)$$
$$\dot{x} = v$$
$$\dot{e} = x_{reference} - x$$

The third state equation allows me to define *e* as the integral of error

$$e = \int \left(x_{reference} - x\right)dt$$

and integral control based on that variable, e.g.

$$F_{source} = K_I e = K_I \int \left(x_{reference} - x\right)dt$$

The "bottom line" is that the number of state variables (and hence the order of the "system of state equations") required for specific analyses may often exceed the number of independent energy storage elements. The "extra" state variables do not determine energy storage. The *minimal* number of state variables is whatever is required to determine energy storage.

This is related to but distinct from the notion of "generalized coordinates" and "generalized velocities". Generalized coordinates are a minimal set of variables sufficient uniquely to define a system's configuration. Clearly, to define the configuration of this system I need one variable (e.g., the position of the mass).

It is common to assume that for mechanical systems the describing differential equations are *second-order* in the generalized coordinates, hence this system would be second order. That may be true if I want to define the position that results from a known force but it is by no means necessary. In some cases (e.g., those in which displacement does not affect energy, as in the example above) I may need fewer differential equations. This is the origin of what are known in classical mechanics as "ignorable" coordinates. In other cases (such as integral control of position, not addressed in classical mechanics, as far as I know) I may need more differential equations.

A further subtlety to which we will return much later is the relation between generalized coordinates and degrees of freedom. Intuitively, "degrees of freedom" describe the ways a system may move. The number of generalized coordinates must at least equal the number of degrees of freedom. However, I may need *more* generalized coordinates than degrees of freedom if the constraints on system motion cannot be expressed as constraints relating the generalized coordinates, i.e., the constraints are *non-holonomic*.

Steering an automobile provides an example: For any given steering wheel position, motion is confined to lie on a circular arc with its center somewhere on an extended axis through the non-steering wheels; thus there's one degree of freedom. However, I need three variables to describe the configuration of the car: lateral and longitudinal position and orientation (with respect to some convenient reference frame).

There is no "proper" choice of state variables as they are not unique, but depending on the system, some are better than others. They should at least be sufficient to determine the total system energy. Assigning both maximal integral causality *and* maximal differential causality to a bond graph will reliably identify the minimal number of state variables required. Which variables to use is a matter of choice, guided by experience.

Strictly speaking, you can *always* use displacements and momenta (respectively) of independent capacitances and inertances (known as *energy* variables or *Hamiltonian* variables). However, that may be a clumsy choice. For electrical networks it would correspond to the charge of independent capacitors and flux-linkage of independent inductors. Most of us have a poor intuitive sense of the physical meaning of flux linkage.

For networks of linear elements, efforts and flows (respectively) of independent capacitances and inertances (often termed *power* variables or *circuit* variables) are usually more convenient and familiar.

For mechanical systems, especially those with nonlinear kinematics, (generalized) displacements and (generalized) velocities (often termed *Lagrangian* variables) are usually easiest to work with.

Neville Hogan
9/27/06