Models with Nonlinear Energy Storage Elements: Energy Variables

If any of the energy storage elements in a model have nonlinear constitutive equations, then power or circuit variables may be a poor choice for the state variables associated with those elements. This is because differentiating a nonlinear constitutive equation (in step 6 above) will not necessarily result in a function only of power variables and their rates of change. For example, consider a nonlinear capacitor:

$$\frac{de}{dt} = \frac{?F}{?q}(q)\,\frac{dq}{dt} = \frac{?F}{?q}(q) \quad f \tag{6.10}$$

Now it is necessary to substitute for the variable q as well as the variable f. One might try to do so by inverting the capacitor constitutive equation.

$$\frac{de}{dt} = \frac{?F}{?q}\left(F^{-1}(e)\right) \quad f \tag{6.11}$$

However, there are two problems with this approach: First of all, the required inverse function may not exist. Secondly, even if it does, the required algebra may be quite tedious. A better alternative is to choose different state variables: the displacements and momenta associated with independent energy storage elements — known as energy state variables or Hamiltonian state variables. Steps 5 and 6 of the substitution procedure are changed as follows.

5. Choose energy state variables. These are the displacements associated with independent capacitors and the momenta associated with independent inertias. The rate of change of each state variable is equal to the <u>input</u> variable to the corresponding independent energy storage element.

   5a. independent capacitor: $dq/dt = f$

   5b. independent inertia: $dp/dt = e$

6. Using its constitutive equation, write the <u>output</u> variable for each independent energy storage element as a function of the corresponding state variable.

   6a. independent capacitor: $e = \Phi(q)$

   6b. independent inertia: $f = \Psi(p)$

The rest of the substitution procedure remains unchanged.

Example: Nonlinear Electric Circuit

Consider the electric circuit of figure 6.1a but assume that the capacitor and inductor have the following nonlinear constitutive equations. For the capacitor:

$$e_C = \frac{1}{C} \frac{q_C q_s}{\sqrt{q_s^2 - q_C^2}} \tag{6.12}$$

where q_s is the saturation charge, (a constant) the maximum charge which may be stored in the capacitor. Note that for $q_C << q_s$ the constitutive equation reduces to that of an ideal (linear) capacitor.

$$\lim_{q_C \Delta 0} e_C = \frac{q_C}{C} \tag{6.13}$$

For the inductor:

$$i_L = \frac{1}{L} \frac{\lambda_L \lambda_s}{\sqrt{\lambda_s^2 - \lambda_L^2}} \tag{6.14}$$

where λ_s is the saturation flux linkage, (a constant) the maximum flux linkge which may stored in the inductor. For $\lambda_L << \lambda_s$ the constitutive equation reduces to that of an ideal (linear) inductor.

$$\lim_{l_L \Delta 0} i_L = \frac{\lambda_L}{L} \tag{6.15}$$

If we were to choose the capacitor voltage as a state variable, differentiating the constitutive equation would result in the following relation.

$$\frac{d}{dt} e_C = \left(\frac{q_s}{C(q_s^2 - q_C^2)^{1/2}} + \frac{q_C^2 q_s}{C(q_s^2 - q_C^2)^{3/2}} \right) i_C \tag{6.16}$$

The nonlinear capacitor equation may be inverted as follows.

$$q_C = \frac{q_s e_C C}{\sqrt{q_s^2 + e_C^2 C^2}} \tag{6.17}$$

Therefore, in this case, the charge q_C may be eliminated from equation 6.16 and the capacitor voltage could be used as a state variable. The resulting state equation is a little intimidating:

$$\frac{d}{dt} e_C = \left(\frac{q_s}{C\left(q_s^2 - \frac{q_s^2 e_C^2 C^2}{q_s^2 + e_C^2 C^2}\right)^{1/2}} + \frac{\left(\frac{q_s^2 e_C^2 C^2}{q_s^2 + e_C^2 C^2}\right) q_s}{C\left(q_s^2 - \frac{q_s^2 e_C^2 C^2}{q_s^2 + e_C^2 C^2}\right)^{3/2}} \right) i_C \tag{6.18}$$

By a similar argument, the inductor current could also be used as a state variable. However, it is *far* simpler to use the capacitor charge and the inductor flux linkages as state variables.

$$\frac{d}{dt} q_C = i_C \tag{6.19}$$

$$\frac{d}{dt} \lambda_L = e_L \tag{6.20}$$

Reading the junction equation from the causal graph (figure 6.1d) $i_C = i_L$. Using the nonlinear inductor constitutive equation (6.14) we obtain one state equation.

$$\frac{dq_C}{dt} = \frac{1}{L} \frac{\lambda_L \lambda_s}{\sqrt{\lambda_s^2 - \lambda_L^2}} \tag{6.21}$$

As before, reading from the causal graph, the inductor voltage is determined by the other three one-port elements

$$e_L = e_S - e_C - e_R \tag{6.22}$$

Substitute for e_C and e_R in equation 6.22 using the constitutive equations of the nonlinear capacitor (6.12) and the resistor (6.6).

$$\frac{d}{dt} \lambda_L = e_S - \frac{1}{C} \frac{q_C q_s}{\sqrt{q_s^2 - q_C^2}} - R\, i_R \tag{6.23}$$

Reading the junction equation from the causal graph, the resistor current is determined by the inductor current $i_R = i_L$. Substituting using the inductor constitutive equation, the second state equation is:

$$\frac{d}{dt} \lambda_L = e_S - \frac{1}{C} \frac{q_C q_s}{\sqrt{q_s^2 - q_C^2}} - R \frac{\lambda_L \lambda_s}{\sqrt{\lambda_s^2 - \lambda_L^2}} \tag{6.24}$$

As before, the substitution process stops when the rate of change of a state variable has been expressed as a function of state and input variables. Note that in this nonlinear system, there is no clear way to express the state equations in vector/matrix form.

Energy variables may also be used for systems composed exclusively of linear elements. For this reason, energy variables have been proposed as the exclusive choice of state variables for systems[1] represented by bond graphs. However, there are several reasons why this is not recommended:

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[1] Karnopp, Dean and Rosenberg, Ronald (1975) System Dynamics: A Unified Approach, John Wiley & Sons, New York.

- physical system behavior is fundamentally independent of the choice of variables, so we should expect no universal rule for choosing state variables.

- energy variables may needlessly complicate the equation derivation process.

- energy variables may be less familiar and less comprehensible than the corresponding power variables.

This last point may seem trivial; in fact, it is probably the most important. One of the primary reasons for developing models is to enhance understanding. For most of us, the current in an inductor is more meaningful than the corresponding flux linkage; the speed of a mass is more readily visualized than its momentum.