Massachusetts Institute of Technology
Department of Mechanical Engineering

2.160 Identification, Estimation, and Learning
Spring 2006

Problem Set No. 7
Out: April 24, 2006 Due: May 3, 2006

Problem 1

Answer the following questions about power spectral density.

1-1 Obtain the power spectrum density function of a sinusoid:

$$u(t) = A\cos\omega_0 t$$

and plot it against frequency ω, $-\infty < \omega < +\infty$.

1.2 A common autocorrelation function encountered in physical problems is

$$R(\tau) = \sigma^2 e^{-\beta|\tau|}\cos\omega_0\tau$$

a). Find the corresponding spectral density function.

b). $R(\tau)$ will be recognized as a damped cosine function. Sketch both the autocorrelation and spectral density functions for the lightly damped case.

1-3 A stationary random process $X(t)$ has a spectral density function of the form:

$$S_X(\omega) = \frac{6\omega^2 + 12}{(\omega^2+4)(\omega^2+1)}$$

What is the mean-square value of $X(t)$? [Hint: Expand $S_X(\omega)$ into two terms: $S_X(\omega) = \dfrac{A}{(\omega^2+4)} + \dfrac{B}{(\omega^2+1)}$, and then integrate each term using standard integral tables.]

Problem 2

Shown below is the cardiovascular system with input *u(t)* being the blood flow rate from the left ventricle and output *y(t)* the blood pressure at the aorta. To identify the system, input-output data, {*u(t)*} and {*y(t)*}, were recorded for a long period of time, and the auto-covariance of the input sequence: $R_u(\tau) = E[u(t)u(t-\tau)]$ was computed for the first several τ ;

$$R_u(0) = 2,\ R_u(1) = 1,\ R_u(2) = -1,\ R_u(3) = -2,\ R_u(4) = -1,\ R_u(5) = 1, \cdots$$

For this system, two model structures for the transfer function from *u(t)* to *y(t)* are considered:

$$M_A(\theta_A) = \{\theta_A = (a, b)^T \mid G_A(q) = \frac{bq^{-1}}{1+aq^{-1}}\},$$
$$M_B(\theta_B) = \{\theta_B = (a_1, a_2, b)^T \mid G_B(q) = \frac{bq^{-1}}{1+a_1q^{-1}+a_2q^{-2}}\}$$

Given the auto-covariance, is the data set informative enough with respect to model structure M_A? Is it informative enough with respect to M_B ? Explain why.

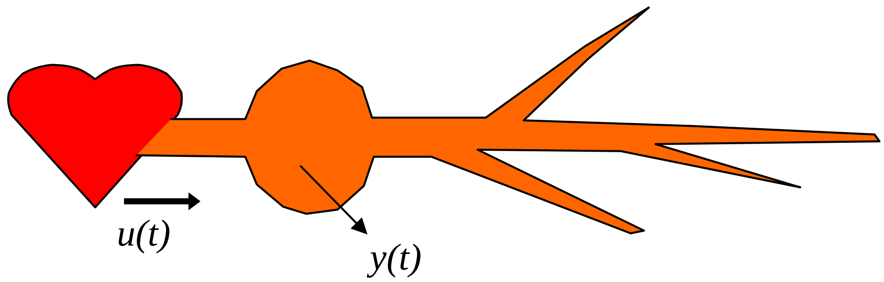

Figure. Schematic diagram of cardiovascular system

Problem 3

Consider the following true system and model structure with parameter vector θ,

$$S: \quad y(t) + 0.5y(t-1) = 2u(t-1) + e_0(t)$$
$$M(\theta): \; y(t) + ay(t-1) = bu(t-1) + e(t) + ce(t-1)$$
$$\theta = (a, c, b)^T$$

where input sequence {*u(t)*} is white noise with variance μ and {$e_0(t)$} is white noise with variance λ. The input {*u(t)*} is uncorrelated with noise {$e_0(t)$} and {*e(t)*}. Answer the following questions.

1). Compute covariances,

$$R_y(0) = E[y^2(t)],\; R_{yu}(0) = E[y(t)u(t)],\; and\; R_{ye}(0) = E[y(t)e_0(t)].$$

[Hint: Read Appendix 2C Covariance Formulas in Ljung’s textbook p.61.]

2). Obtain the asymptotic variance of parameter estimate: $\hat{\theta}_N = [\hat{a}, \hat{c}, \hat{b}]^T$, when a quadratic prediction-error criterion is used.

Problem 4

[Read Example 9.1 of Ljung's textbook first before solving the following problem.]

Assume that the following ARMAX model structure is used in Example 9.1,

$$M(\theta): \quad y(t) + a\,y(t-1) = u(t-1) + e(t) + c\,e(t-1)$$
$$\theta = [a, c]^T$$

Let the parameter vector θ be estimated by a quadratic-error method. Show that:

$$Cov\ \hat{a}_N \sim \frac{\lambda_0}{N}\frac{1-a_0^2}{\mu + \lambda_0 a_0^2}\,.$$