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2.161 Signal Processing: Continuous and Discrete
Fall 2008

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Lecture 6¹

Reading:

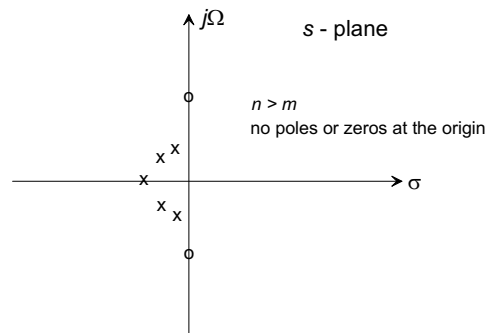
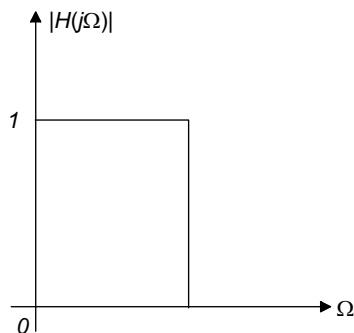
- Class handout: *Sinusoidal Frequency Response of Linear Systems* (Sec. 7).
- Class handout: *Sinusoidal Frequency Response of Linear Systems* (Sec. 6.1).
- Class handout: *Introduction to Continuous Time Filter Design*.

1 Poles and Zeros of Filter Classes

We can use the s -plane relationship between a filter's poles and zeros and the frequency response function to make some general comments about the design of various classes of filters:

(a) Low-Pass Filters:

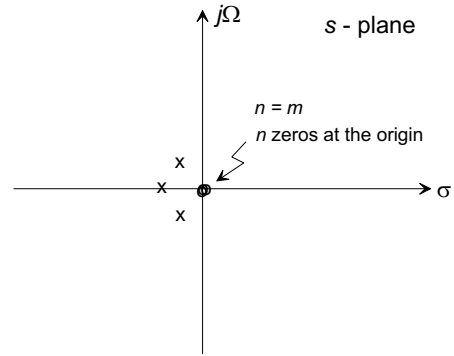
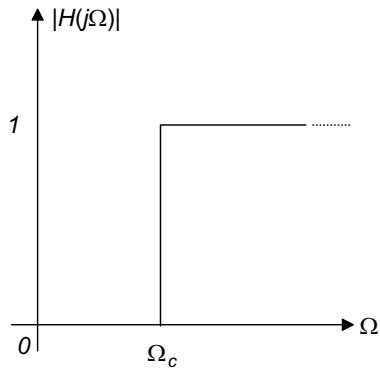
- To ensure a high-frequency roll-off the number of poles must exceed the number of zeros, ie $n > m$. (In many low-pass filters $m = 0$.)
- To ensure a finite low frequency gain there can be no poles or zeros at the origin.



(b) High-Pass Filters

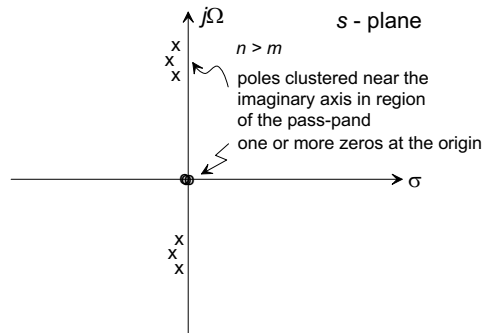
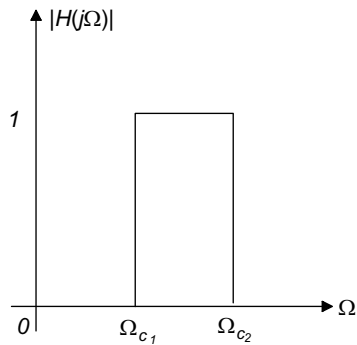
- To ensure a constant high-frequency gain the number of poles must equal the number of zeros, ie $n = m$.
- To ensure a low frequency gain that approaches zero, there must be one or more zeros at the origin.

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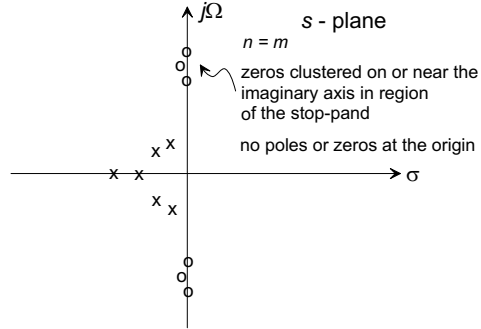
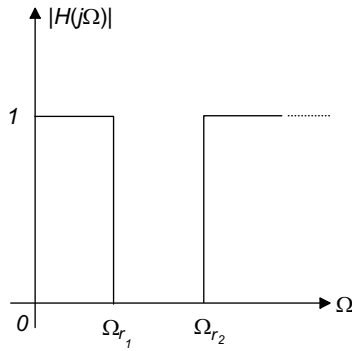
(c) Band-Pass Filters

- To ensure a high-frequency roll-off the number of poles must exceed the number of zeros, ie $n > m$.
- To ensure a low frequency gain that approaches zero, there must be one or more zeros at the origin.
- The band-pass characteristic is shaped by a group of poles clustered near the imaginary axis in the region of the passband,



(d) Band-Stop Filters

- To ensure a constant high-frequency gain the number of poles must equal the number of zeros, ie $n = m$.
- To ensure a finite low frequency gain there can be no poles or zeros at the origin.
- The band-reject characteristic is shaped by a group of zeros clustered on or near the imaginary axis in the region of the stopband,



2 The Decibel

Filter frequency response magnitudes $|H(j\Omega)|$ are frequently plotted using the decibel logarithmic scale. The Bel, named after Alexander Graham Bell, is defined as the logarithm to base 10 of the *ratio* of two *power* levels. In practice the Bel is too large a unit, and the decibel (abbreviated dB), defined to be one tenth of a Bel, has become the standard unit of logarithmic power ratio. The power flow \mathcal{P} into any element in a system, may be expressed in terms of a logarithmic ratio Q to a *reference* power level \mathcal{P}_{ref} :

$$Q = \log_{10} \left(\frac{\mathcal{P}}{\mathcal{P}_{ref}} \right) \text{ Bel} \quad \text{or} \quad Q = 10 \log_{10} \left(\frac{\mathcal{P}}{\mathcal{P}_{ref}} \right) \text{ dB.} \quad (1)$$

Because the power dissipated in a D-type element is proportional to the square of the amplitude of a system variable applied to it, when the ratio of across or through variables is computed the definition becomes

$$Q = 10 \log_{10} \left(\frac{A}{A_{ref}} \right)^2 = 20 \log_{10} \left(\frac{A}{A_{ref}} \right) \text{ dB.} \quad (2)$$

where A and A_{ref} are amplitudes of variables. ² Table ?? expresses some commonly used decibel values in terms of the power and amplitude ratios.

The magnitude of the frequency response function $|H(j\Omega)|$ is defined as the *ratio* of the amplitude of a sinusoidal output variable to the amplitude of a sinusoidal input variable. This ratio is expressed in decibels, that is

$$20 \log_{10} |H(j\Omega)| = 20 \log_{10} \frac{|Y(j\Omega)|}{|U(j\Omega)|} \text{ dB.}$$

As noted this usage is not strictly correct because the frequency response function does not define a power ratio, and the decibel is a dimensionless unit whereas $|H(j\Omega)|$ may have physical units.

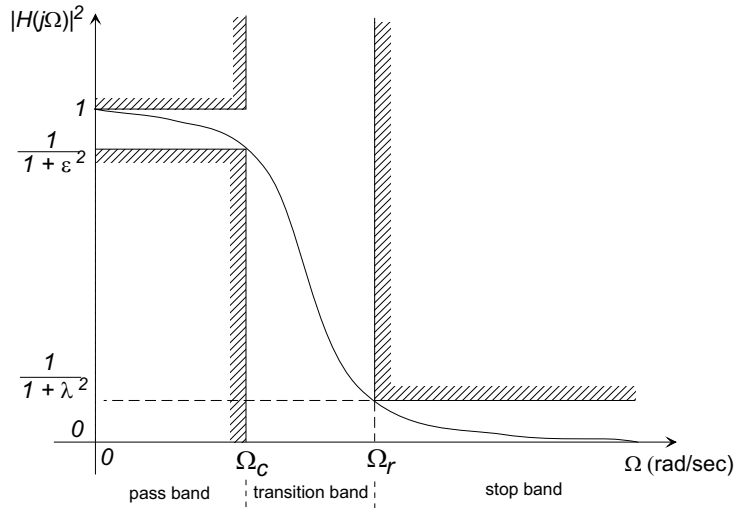
²This definition is only strictly correct when the two amplitude quantities are measured across a common D-type (dissipative) element. Through common usage, however, the decibel has been effectively redefined to be simply a convenient logarithmic measure of amplitude ratio of any two variables. This practice is widespread in texts and references on system dynamics and control system theory. In this book we have also adopted this convention.

Decibels	Power Ratio	Amplitude Ratio
-40	0.0001	0.01
-20	0.01	0.1
-10	0.1	0.3162
-6	0.25	0.5
-3	0.5	0.7071
0	1.0	1.0
3	2.0	1.414
6	4.0	2.0
10	10.0	3.162
20	100.0	10.0
40	10000.0	100.0

Table 1: Common Decibel quantities and their corresponding power and amplitude ratios.

3 Low-Pass Filter Design

The prototype low-pass filter is based upon the magnitude-squared of the frequency response function $|H(j\Omega)|^2$, or the frequency response power function. The phase response of the filter is not considered. We begin by defining tolerance regions on the power frequency response plot, as shown below:



The filter specifications are that

$$1 \geq |H(j\Omega)|^2 > \frac{1}{1+\epsilon^2} \quad \text{for } |\Omega| < \Omega_c$$

and

$$|H(j\Omega)|^2 < \frac{1}{1+\lambda^2} \quad \text{for } |\Omega| > \Omega_r,$$

where Ω_c is the cut-off frequency, Ω_r is the rejection frequency, and ϵ and λ are design parameters that select the filter attenuation at the two critical frequencies. For example, if

$\epsilon = 1$, at Ω_c the power response $|H(j\Omega_c)|^2 = 0.5$, the -3 dB response frequency. In general we expect the response function to be monotonically decreasing in the transition band.

The filter functions examined in this document will be of the form

$$|H(j\Omega)|^2 = \frac{1}{1 + f^2(\Omega)}. \quad (3)$$

where $f(\Omega) \rightarrow 0$ as $\Omega \rightarrow 0$, and $f(\Omega) \rightarrow \infty$ as $\Omega \rightarrow \infty$ to generate a low-pass filter action.

3.1 The Butterworth Filter

The Butterworth filter is defined by the power gain

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 (\Omega/\Omega_c)^{2N}}$$

where N is a positive integer defining the filter order. Note that λ does not appear in this formulation, but clearly N and λ are interrelated, since at $\Omega = \Omega_r$

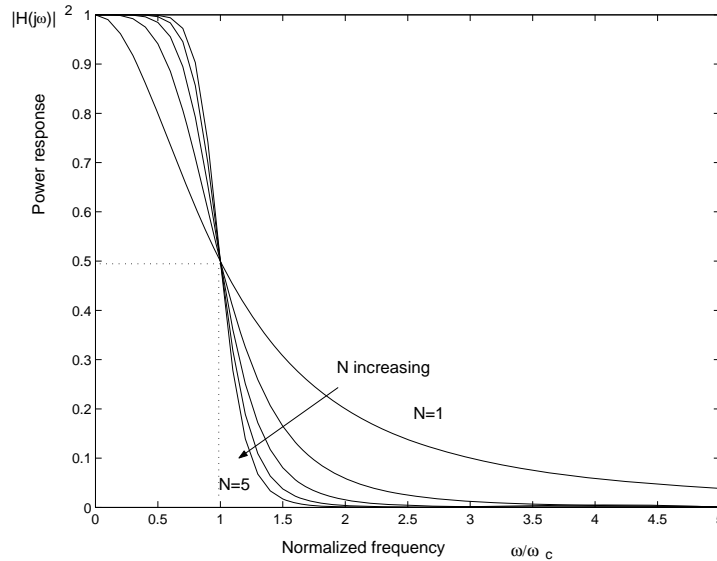
$$\frac{1}{1 + \lambda^2} \geq \frac{1}{1 + \epsilon^2 (\Omega_r/\Omega_c)^{2N}}$$

which may be solved to show

$$N \geq \frac{\log(\lambda/\epsilon)}{\log(\Omega_r/\Omega_c)}$$

endequation

The power response function of Butterworth filters for $N = 1 \dots 5$ is shown below:



Butterworth filters are also known as “maximally flat” filters because the response has the maximum number of vanishing derivatives at $\Omega = 0$ and $\Omega = \infty$ for filters of the form of Eq. 1.

3.1.1 The Butterworth Filter Transfer Function

The poles of the power gain transfer function may be found by substituting $s = j\Omega$ in the characteristic equation:

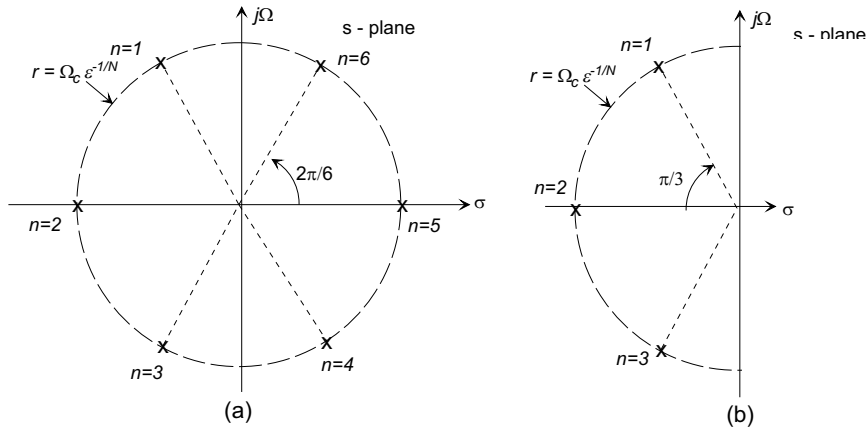
$$1 + \epsilon^2 \left(\frac{s}{j\Omega_c} \right)^{2N} = 0$$

and solving for S , which yields $2N$ roots p_n ($n = 1 \dots 2N$) that lie on a circle with radius $r = \Omega_c \epsilon^{-1/2N}$, and angular separation of $\pi/2N$ rad:

$$p_n = \Omega_c \epsilon^{-1/2N} e^{j\pi(2n+N-1)/2N} \quad n = 1 \dots 2N$$

Notice that if N is odd a pair of poles will lie on the real axis at $s = \pm \Omega_c \epsilon^{-1/2N}$, while if N is even the roots will form complex conjugate pairs.

The six poles of $|H(s)|^2$ for a third-order ($N = 3$) Butterworth filter are shown in (a) below:



For a stable causal system we must have

$$|H(j\Omega)|^2 = H(j\Omega)\overline{H(j\Omega)} = H(s)H(-s)|_{s=j\Omega}$$

with all poles in the left-half-plane, which allows us to take the N poles of $|H(j\Omega)|^2$ in the left half-plane as the poles of the filter $H(s)$, that is the poles specified by ($n = 1 \dots N$) in Eq. (8) above,

$$p_n = \Omega_c \epsilon^{-1/2N} e^{j\pi(2n+N-1)/2N} \quad n = 1 \dots N \quad (4)$$

as is shown in (b) above.

If the filter is to have unity gain at low frequencies

$$\lim_{\Omega \rightarrow 0} |H(j\Omega)| = 1$$

we require the complete Butterworth transfer function to be

$$\begin{aligned} H(s) &= \frac{(-p_1)(-p_2) \dots (-p_N)}{(s - p_1)(s - p_2) \dots (s - p_N)} \\ &= \frac{(-1)^N p_1 p_2 \dots p_N}{(s - p_1)(s - p_2) \dots (s - p_N)}, \end{aligned}$$

where only the N stable left-half-plane poles (Eq. 4) are included.

Design Procedure:

1. Determine the filter order

$$N \geq \frac{\log(\lambda/\epsilon)}{\log(\Omega_r/\Omega_c)}$$

2. Determine the Butterworth radius

$$r = \Omega_c \epsilon^{-1/N}$$

3. Determine The Butterworth angles

$$\theta_n = \pi \frac{2n + N - 1}{2N} \quad n = 1 \dots 2N$$

4. Determine the N left half-plane poles

$$p_n = r e^{j\theta_n} \quad n = 1, \dots, N$$

5. Form the transfer function

$$H(s) = \frac{-p_1 p_2 \dots p_N}{(s - p_1)(s - p_2) \dots (s - p_N)}$$