2.161 Signal Processing: Continuous and Discrete
Fall 2008

Introduction to Frequency Domain Processing [1]

1 Introduction

In this set of notes we examine an alternative to the time-domain convolution operations describing the input-output operations of a linear processing system. The methods developed here use *Fourier techniques* to transform the temporal representation of a waveform, $x(t)$, to a frequency domain representation, $X(j\Omega)$, where the operation of *convolution* is replaced by simple *multiplication*. In addition, an understanding of Fourier methods gives qualitative insights to signal processing techniques, such as filtering.

The methods are known collectively as *Fourier analysis* methods, after Jean Baptiste Joseph Fourier, who in the early part of the 19th century proposed that an arbitrary repetitive function could be written as an infinite sum of sine and cosine functions [1]. The *Fourier series* representation of periodic functions may be extended through the *Fourier transform* to represent non-repeating aperiodic (or transient) functions as a *continuous distribution* of sinusoidal components. A further generalization produces the *Laplace transform* representation of waveforms.

The methods of representing and analyzing waveforms and system responses in terms of the action of the frequency response function on component sinusoidal or exponential waveforms are known collectively as the *frequency-domain* methods. Such methods, developed in this chapter, have important theoretical and practical applications throughout engineering, especially in signal processing, system dynamics and control system theory.

2 Fourier Analysis of Periodic Waveforms

Consider the steady-state response of linear time-invariant systems to two *periodic* waveforms, the real sinusoid $x(t) = \sin\Omega t$, and the complex exponential $x(t) = e^{j\Omega t} = \cos\Omega t + j\sin\Omega t$. Both functions are repetitive; that each of them has repeating values at intervals of $t = 2\pi/\Omega$ seconds apart. In general, a *periodic function* is a function that satisfies the relationship:

$$x(t) = x(t+T)$$

for all t, or $x(t) = x(t+nT)$ for $n = \pm 1, \pm 2, \pm 3, \ldots$, where T is defined as the *period*. Figure 1 shows some examples of periodic functions.

The *fundamental angular frequency* Ω_0 (in radians/second) of a periodic waveform is defined directly from the period

$$\Omega_0 = \frac{2\pi}{T},$$

and the *fundamental frequency* F_0 (in Hz) is

$$F_0 = \frac{1}{T}$$

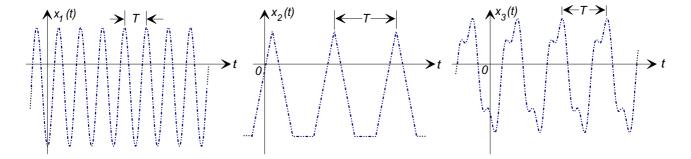---

[1]D. Rowell, September 16, 2008

Figure 1: Examples of periodic functions of time.

so that $\Omega_0 = 2\pi F_0$.

Any periodic function with period T is also be periodic at intervals of nT for any positive integer n. Similarly any waveform with a period of T/n is periodic at intervals of T seconds. Two waveforms whose periods, or frequencies, are related by a simple integer ratio are said to be *harmonically related*.

Consider, for example, a pair of periodic functions; the first $x_1(t)$ with a period of $T_1 = 12$ seconds, and the second $x_2(t)$ with a period of $T_2 = 4$ seconds. If the fundamental frequency Ω_0 is defined by $x_1(t)$, that is $\Omega_o = 2\pi/12$, then $x_2(t)$ has a frequency of $3\Omega_0$. The two functions are harmonically related, and $x_2(t)$ is said to have a frequency which is the *third* harmonic of the fundamental Ω_0. If these two functions are summed together to produce a new function $g(t) = x_1(t) + x_2(t)$, then $g(t)$ will repeat at intervals defined by the longest period of the two, in this case every 12 seconds. In general, when harmonically related waveforms are added together the resulting function is also periodic with a repetition period equal to the fundamental period.

■ Example 1

A family of waveforms $g_N(t)$ ($N = 1, 2 \dots 5$) is formed by adding together the first N of up to five component functions, that is

$$g_N(t) = \sum_{n=1}^{N} x_n(t) \qquad 1 < N \leq 5$$

where

$$
\begin{aligned}
x_1(t) &= 1 \\
x_2(t) &= sin(2\pi t) \\
x_3(t) &= \frac{1}{3} sin(6\pi t) \\
x_4(t) &= \frac{1}{5} sin(10\pi t) \\
x_5(t) &= \frac{1}{7} sin(14\pi t).
\end{aligned}
$$

The first term is a constant, and the four sinusoidal components are harmonically related, with a fundamental frequency of $\Omega_0 = 2\pi$ rad/s and a fundamental period of $T = 2\pi/\Omega_0 = 1$ second. (The constant term may be considered to be periodic with any
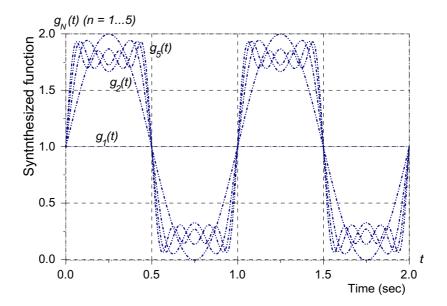
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Figure 2: Synthesis of a periodic waveform by the summation of harmonically related components

arbitrary period, but is commonly considered to have a frequency of zero rad/s.) Figure 2 shows the evolution of the function that is formed as more of the individual terms are included into the summation. Notice that in all cases the period of the resulting $g_N(t)$ remains constant and equal to the period of the fundamental component (1 second). In this particular case, it can be seen that the sum is tending toward a square wave as more terms are included.

Similarly, two sinusoidal components with angular frequencies of 10 rad/s and 15 rad/s are each harmonics of the fundamental frequency $\Omega_0 = 5$ rad/s. If added together, these two components would generate a waveform with a period $T = 2\pi/5$ seconds, even though there is no component at the fundamental frequency.

The *Fourier series* [2,3] representation of a real periodic function $x(t)$ is based upon the summation of harmonically related sinusoidal components. If the period is T, then the harmonics are sinusoids with frequencies that are integer multiples of Ω_0, that is the nth harmonic component has a frequency $n\Omega_0 = 2\pi n/T$, and can be written

$$
\begin{aligned}
x_n(t) &= a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t) & (1) \\
&= \mathcal{A}_n \sin(n\Omega_0 t + \phi_n). & (2)
\end{aligned}
$$

In the first form the function $x_n(t)$ is written as a pair of sine and cosine functions with real coefficients a_n and b_n. The second form, in which the component is expressed as a single sinusoid with an amplitude $\mathcal{A}_n$ and a phase ϕ_n, is directly related to the first by the trigonometric relationship:

$$
\mathcal{A}_n \sin(n\Omega_0 t + \phi_n) = \mathcal{A}_n \sin\phi_n \cos(n\Omega_0 t) + \mathcal{A}_n \cos\phi_n \sin(n\Omega_0 t).
$$

Equating coefficients,

$$
\begin{aligned}
a_n &= \mathcal{A}_n \sin\phi_n \\
b_n &= \mathcal{A}_n \cos\phi_n & (3)
\end{aligned}
$$

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and

$$\begin{aligned}
\mathcal{A}_n &= \sqrt{a_n^2 + b_n^2} \\
\phi_n &= \tan^{-1}(a_n/b_n).
\end{aligned} \tag{4}$$

The Fourier series representation of an arbitrary periodic waveform $x(t)$ (subject to some general conditions described later) is as an infinite sum of harmonically related sinusoidal components, commonly written in the following two equivalent forms

$$\begin{aligned}
x(t) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t) \right) \tag{5} \\
&= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \mathcal{A}_n \sin(n\Omega_0 t + \phi_n). \tag{6}
\end{aligned}$$

In either representation knowledge of the fundamental frequency Ω_0, and the sets of Fourier coefficients $\{a_n\}$ and $\{b_n\}$ (or $\{\mathcal{A}_n\}$ and $\{\phi_n\}$) is sufficient to completely define the waveform $x(t)$.

A third, and completely equivalent, representation of the Fourier series expresses each of the harmonic components $x_n(t)$ in terms of complex exponentials instead of real sinusoids. The Euler formulas may be used to replace each sine and cosine terms in the components of Eq. (1) by a pair of complex exponentials

$$\begin{aligned}
x_n(t) &= a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t) \\
&= \frac{a_n}{2} \left(e^{jn\Omega_0 t} + e^{-jn\Omega_0 t} \right) + \frac{b_n}{2j} \left(e^{jn\Omega_0 t} - e^{-jn\Omega_0 t} \right) \\
&= \frac{1}{2}(a_n - jb_n) e^{jn\Omega_0 t} + \frac{1}{2}(a_n + jb_n) e^{-jn\Omega_0 t} \\
&= X_n e^{jn\Omega_0 t} + X_{-n} e^{-jn\Omega_0 t} \tag{7}
\end{aligned}$$

where the new coefficients

$$\begin{aligned}
X_n &= 1/2(a_n - jb_n) \\
X_{-n} &= 1/2(a_n + jb_n) \tag{8}
\end{aligned}$$

are now complex numbers. With this substitution the Fourier series may be written in a compact form based upon harmonically related complex exponentials

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\Omega_0 t}. \tag{9}$$

This form of the series requires summation over all negative and positive values of n, where the coefficients of terms for positive and negative values of n are complex conjugates,

$$X_{-n} = \overline{X}_n, \tag{10}$$

so that knowledge of the coefficients X_n for $n \geq 0$ is sufficient to define the function $x(t)$.

Throughout these notes we adopt the nomenclature of using upper case letters to represent the Fourier coefficients in the *complex* series notation, so that the set of coefficients $\{G_n\}$ represent the function $g(t)$, and $\{Y_n\}$ are the coefficients of the function $y(t)$. The lower case coefficients $\{a_n\}$ and $\{b_n\}$ are used to represent the *real* Fourier coefficients of any function of time.

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■ Example 2

A periodic function $x(t)$ consists of five components

$$x(t) = 2 + 3\sin(100t) + 4\cos(100t) + 5\sin(200t + \pi/4) + 3\cos(400t).$$

It may be expressed as a finite complex Fourier series by expanding each term through the Euler formulas

$$
\begin{aligned}
x(t) &= 2 + \frac{3}{2j}\left(e^{j100t} - e^{-j100t}\right) + \frac{4}{2}\left(e^{j100t} + e^{-j100t}\right) \\
&\quad + \frac{5}{2j}\left(e^{j(200t+\pi/4)} - e^{-j(200t+\pi/4)}\right) + \frac{3}{2}\left(e^{j400t} + e^{-j400t}\right) \\
&= 2 + \left(2 + \frac{3}{2j}\right)e^{j100t} + \left(2 - \frac{3}{2j}\right)e^{-j100t} \\
&\quad \left(\frac{5}{2\sqrt{2}} + \frac{5}{2\sqrt{2}j}\right)e^{j200t} + \left(\frac{5}{2\sqrt{2}} - \frac{5}{2\sqrt{2}j}\right)e^{-j200t} \\
&\quad + \frac{3}{2}e^{j400t} + \frac{3}{2}e^{-j400t}.
\end{aligned}
$$

The fundamental frequency is $\Omega_0 = 100$ radians/second, and the time-domain function contains harmonics $n = 1, 2, 3,$ and 4. The complex Fourier coefficients are

$$
\begin{aligned}
X_0 &= 2 \\
X_1 &= 2 - \frac{3}{2}j & X_{-1} &= 2 + \frac{3}{2}j \\
X_2 &= \frac{5}{2\sqrt{2}}(1 - 1j) & X_{-2} &= \frac{5}{2\sqrt{2}}(1 + 1j) \\
X_3 &= 0 & X_{-3} &= 0 \\
X_4 &= \frac{3}{2} & X_{-4} &= \frac{3}{2}
\end{aligned}
$$

The finite Fourier series may be written in the complex form using these coefficients as

$$x(t) = \sum_{n=-5}^{5} X_n e^{jn100t}$$

and plotted with real and imaginary parts as in Fig. 3.

The values of the Fourier coefficients, in any of the three above forms, are effectively measures of the amplitude and phase of the harmonic component at a frequency of $n\Omega_0$. The *spectrum* of a periodic waveform is the set of all of the Fourier coefficients, for example $\{\mathcal{A}_n\}$ and $\{\phi_n\}$, expressed as a function of frequency. Because the harmonic components exist at discrete frequencies, periodic functions are said to exhibit *line spectra*, and it is common to express the spectrum graphically with frequency Ω as the independent axis, and with the Fourier coefficients plotted as lines at intervals of Ω_0. The first two forms of the Fourier series, based upon Eqs. (1) and (2), generate "one-sided" spectra because they are defined from positive values of n only, whereas the complex form defined by Eq. (9) generates a "two-sided" spectrum because its summation requires positive and negative values of n. Figure 3 shows the complex spectrum for the finite series discussed in Example 2.
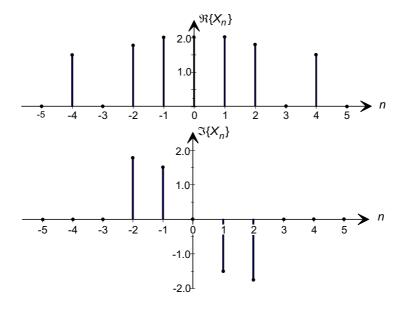
Figure 3: Spectral representation of the waveform discussed in Example 2.

2.1 Computation of the Fourier Coefficients

Assume that a periodic waveform $x_p(t)$, with period T, is described by a set of K discrete samples $\{x_k\} = \{x(k\Delta)\}$, taken at equi-spaced intervals Δ over one period, so that $\Delta = T/K$. This waveform is to be *approximated* by a set of values $\{\tilde{x}_k\}$, each of which is a finite sum of N harmonically related components with fundamental angular frequency $\Omega_0 = 2\pi/T$, that is

$$
\begin{aligned}
\tilde{x}_k &= \frac{1}{2}a_0 + \sum_{n=1}^{N-1} a_n \cos(n\Omega_0 k\Delta) + b_n \sin(n\Omega_0 k\Delta) \\
&= \sum_{n=-(N-1)}^{N-1} X_n e^{jnk\Omega_0\Delta}
\end{aligned}
\tag{11}
$$

where the coefficients a_n and b_n, (or X_n), are to be determined. As noted previously

$$
X_n = \frac{1}{2}\left(a_n - jb_n\right)
$$

and $X_{-n} = \bar{F}_n$, so that it is necessary to determine N coefficients in either representation.

We will use the complex exponential representation here. Our task is to find the set of coefficients $\{X_n\}$ that will minimize the *total squared error* $E^2\left(\{X_n\}\right)$ defined as

$$
E^2\left(\{X_n\}\right) = \sum_{k=0}^{K-1} [x_k - \tilde{x}_k]^2 = \sum_{k=0}^{K-1} \left[x_k - \sum_{n=-(N-1)}^{N-1} X_n e^{jkn\Omega_0\Delta}\right]^2.
\tag{12}
$$

The minimum is found by setting the partial derivatives of E^2 with respect to X_m equal to zero for all $m = 0 \ldots N-1$:

$$
\frac{\partial E^2}{\partial X_m} = -2\sum_{k=0}^{K-1} e^{jkm\Omega_0\Delta} \left[x_k - \sum_{n=-(N-1)}^{N-1} X_m e^{jnk\Omega_0\Delta}\right] = 0
\tag{13}
$$

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or

$$\sum_{n=-(N-1)}^{N-1} X_m \sum_{k=0}^{K-1} e^{jk(n+m)\Omega_0\Delta} = \sum_{k=0}^{K-1} x_k e^{jkn\Omega_0\Delta} \qquad (14)$$

The inner sum in Eq. (14) is a geometric series and because the summation is over an integral number of periods,

$$\sum_{n=0}^{N-1} e^{jk(n+m)\Omega_0\Delta} = N\delta_{-n,m}$$

where $\delta_{-n,m}$ is the Kronecker delta. Then Eq. (14) becomes

$$KX_{-n} = \sum_{k=0}^{K-1} x_n e^{jkn\Omega_0\Delta}. \qquad (15)$$

But as noted above $X_{-n} = \bar{X}_n$, and since $\{x_k\}$ are real we have

$$X_n = \frac{1}{K} \sum_{k=0}^{K-1} x_k e^{-jkn\Omega_0\Delta}. \qquad (16)$$

Equation (16) expresses the finite Fourier series as a least-squares approximation to a finite data set with coefficients $\{X_n\}$. However, we can now let the number of samples K, taken over the fundamental period T, become very large, and write $\Delta = T/K = dt$, $k\Delta = t$, and $x_k = x_p(t)$. Then in the limit

$$X_n = \frac{1}{T} \int_0^T x_p(t)e^{-jn\Omega_0 t}dt = \frac{1}{T} \int_{t_0}^{t_0+T} x_p(t)e^{-jn\Omega_0 t}dt \qquad (17)$$

where t_0 is any arbitrary time origin.

The corresponding formulas for the sinusoidal forms of the series may be derived directly from Eq. (17). From Eq. (8) it can be seen that

$$
\begin{aligned}
a_n &= X_n + X_{-n} \\
&= \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \left[e^{jn\Omega_0 t} + e^{-jn\Omega_0 t} \right] dt \\
&= \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos(n\Omega_0 t)dt \qquad (18)
\end{aligned}
$$

and similarly

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin(n\Omega_0 t)dt. \qquad (19)$$

Equations (17), (18), and (19) define the Fourier coefficients $\{X_n\}$ (or $\{a_n\}$ and $\{b_n\}$) as being optimal in a least-squares sense. Note that the values of the coefficients are independent of the number of terms N in the series. The complete Fourier series is defined as N approaches infinity.

The calculation of the coefficients for a given periodic time function $x(t)$ is known as Fourier *analysis* or *decomposition* because it implies that the waveform can be "decomposed" into its spectral components. On the other hand, the expressions that express $x(t)$ as a Fourier series summation (Eqs. (5), (6), and (9)) are termed Fourier *synthesis* equations because the imply that $x(t)$ could be created (synthesized) from an infinite set of harmonically related oscillators.

Table 1 summarizes the analysis and synthesis equations for the sinusoidal and complex exponential formulations of the Fourier series.

7

	Sinusoidal formulation	Exponential formulation
Synthesis:	$x(t) = \dfrac{1}{2}a_0 + \displaystyle\sum_{n=1}^{\infty}\left(a_n\cos(n\Omega_0 t) + b_n\sin(n\Omega_0 t)\right)$	$x(t) = \displaystyle\sum_{n=-\infty}^{+\infty} X_n e^{jn\Omega_0 t}$
Analysis:	$a_n = \dfrac{2}{T}\displaystyle\int_{t_1}^{t_1+T} x(t)\cos(n\Omega_0 t)dt$ $b_n = \dfrac{2}{T}\displaystyle\int_{t_1}^{t_1+T} x(t)\sin(n\Omega_0 t)dt$	$X_n = \dfrac{1}{T}\displaystyle\int_{t_1}^{t_1+T} x(t)e^{-jn\Omega_0 t}dt$

Table 1: Summary of analysis and synthesis equations for Fourier analysis and synthesis.

■ **Example 3**

Find the complex and real Fourier series representations of the periodic square wave $x(t)$ with period T,

$$x(t) = \begin{cases} 1 & 0 \le t < T/2, \\ 0 & T/2 \le t < T \end{cases}$$
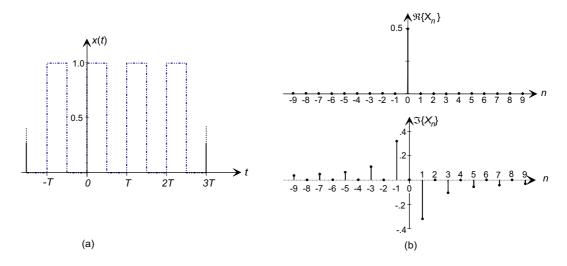
as shown in Fig. 4.

Figure 4: A periodic square wave and its spectrum.

Solution: The complex Fourier series is defined by the synthesis equation

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n e^{jn\Omega_0 t}. \tag{i}$$

In this case the function is non-zero for only half of the period, and the integration limits can be restricted to this range. The zero frequency coefficient X_0 must be computed

separately:

$$X_0 = \frac{1}{T} \int_0^{T/2} e^{j0} dt = \frac{1}{2},$$ (ii)

and all of the other coefficients are:

$$
\begin{aligned}
X_n &= \frac{1}{T} \int_0^{T/2} (1)e^{-jn\Omega_0 t} dt \\
&= \frac{-1}{jn\Omega_0 T} \left[e^{-jn\Omega_0 t} \Big|_0^{T/2} \right] \\
&= \frac{j}{2n\pi} \left[e^{-jn\pi} - 1 \right]
\end{aligned}
$$ (iii)

since $\Omega_0 T = 2\pi$. Because $e^{-jn\pi} = -1$ when n is odd and $+1$ when n is even,

$$
X_n = \begin{cases} -j/n\pi, & \text{if n is odd;} \\ 0, & \text{if n is even.} \end{cases}
$$

The square wave can then be written as the complex Fourier series

$$x(t) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{j}{(2m-1)\pi} \left(e^{-j(2m-1)\Omega_0 t} - e^{j(2m-1)\Omega_0 t} \right).$$ (iv)

where the terms $X_n e^{j\Omega_n t}$ and $X_{-n} e^{-j\Omega_n t}$ have been combined in the summation.

If the Euler formulas are be used to expand the complex exponentials, the cosine terms cancel, and the resulting series involves only sine terms:

$$
\begin{aligned}
x(t) &= \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2}{(2m-1)\pi} \sin\left((2m-1)\Omega_0 t\right) \\
&= \frac{1}{2} + \frac{2}{\pi} \left(\sin(\Omega_0 t) + \frac{1}{3}\sin(3\Omega_0 t) + \frac{1}{5}\sin(5\Omega_0 t) + \frac{1}{7}\sin(7\Omega_0 t) + \ldots \right).
\end{aligned}
$$ (v)

Comparison of the terms in this series with the components of the waveform synthesized in Example 1, and shown in Fig. 2, shows how a square wave may be progressively approximated by a finite series.

2.2 Properties of the Fourier Series

A full discussion of the properties of the Fourier series is beyond the scope of this book, and the interested reader is referred to the references. Some of the more important properties are summarized below.

(1) **Existence of the Fourier Series** For the series to exist, the integral of Eq. (**??**) must converge. A set of three sufficient conditions, known as the Dirichlet conditions, guarantee the existence of a Fourier series for a given periodic waveform $x(t)$. They are

- The function x(t) must be *absolutely integrable* over any period, that is

$$\int_{t_0}^{t_0+T} |x(t)| \, dt < \infty$$ (20)

for any t_0.

- There must be at most a finite number of maxima and minima in the function $x(t)$ within any period.
- There must be at most a finite number of discontinuities in the function $x(t)$ within any period, and all such discontinuities must be finite in magnitude.

These requirements are satisfied by almost all waveforms found in engineering practice. The Dirichlet conditions are a *sufficient* set of conditions to guarantee the existence of a Fourier series representation. They are not *necessary* conditions, and there are some functions that have a Fourier series representation without satisfying all three conditions.

(2) **Linearity of the Fourier Series Representation** The Fourier analysis and synthesis operations are linear. Consider two periodic functions $g(t)$ and $h(t)$ with identical periods T, and their complex Fourier coefficients

$$G_n = \frac{1}{T} \int_0^T g(t) e^{-jn\Omega_0 t} dt$$

$$H_n = \frac{1}{T} \int_0^T h(t) e^{-jn\Omega_0 t} dt$$

and a third function defined as a weighted sum of $g(t)$ and $h(t)$

$$x(t) = ag(t) + bh(t)$$

where a and b are constants. The linearity property, which may be shown by direct substitution into the integral, states that the Fourier coefficients of $x(t)$ are

$$X_n = aG_n + bH_n,$$

that is the Fourier series of a weighted sum of two time-domain functions is the weighted sum of the individual series.

(3) **Even and Odd Functions** If $x(t)$ exhibits symmetry about the $t = 0$ axis the Fourier series representation may be simplified. If $x(t)$ is an even function of time, that is $x(-t) = x(t)$, the complex Fourier series has coefficients X_n that are purely real, with the result that the real series contains only cosine terms, so that Eq. (5) simplifies to

$$x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\Omega_0 t). \tag{21}$$

Similarly if $x(t)$ is an odd function of time, that is $x(-t) = -x(t)$, the coefficients X_n are imaginary, and the one-sided series consists of only sine terms:

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\Omega_0 t). \tag{22}$$

Notice that an odd function requires that $x(t)$ have a zero average value.

(4) **The Fourier Series of a Time Shifted Function** If the periodic function $x(t)$ has a Fourier series with complex coefficients X_n, the series representing a "time-shifted" version $g(t) = x(t + \tau)$ has coefficients $e^{-jn\Omega_0 \tau} X_n$. If

$$X_n = \frac{1}{T} \int_0^T x(t) e^{-jn\Omega_0 t} dt$$

then
$$G_n = \frac{1}{T} \int_0^T f(t+\tau)e^{-jn\Omega_0 t}dt.$$

Changing the variable of integration $\nu = t + \tau$ gives

$$\begin{aligned}
G_n &= \frac{1}{T} \int_\tau^{\tau+T} f(\nu)e^{-jn\Omega_0(\nu-\tau)}d\nu \\
&= e^{jn\Omega_0\tau}\frac{1}{T} \int_\tau^{\tau+T} f(\nu)e^{-jn\Omega_0\nu t}d\nu \\
&= e^{jn\Omega_0\tau} X_n.
\end{aligned}$$

If the nth spectral component is written in terms of its magnitude and phase

$$x_n(t) = \mathcal{A}_n \sin(n\Omega_0 t + \phi_n)$$

then

$$\begin{aligned}
x_n(t+\tau) &= \mathcal{A}_n \sin\left(n\Omega_0(t+\tau) + \phi_n\right) \\
&= \mathcal{A}_n \sin\left(n\Omega_0 t + \phi_n + n\Omega_0\tau\right).
\end{aligned}$$

The additional phase shift $n\Omega_0\tau$, caused by the time shift τ, is directly proportional to the frequency of the component $n\Omega_0$.

(5) Interpretation of the Zero Frequency Term The coefficients X_0 in the complex series and a_0 in the real series are somewhat different from all of the other terms for they correspond to a harmonic component with zero frequency. The complex analysis equation shows that

$$X_0 = \frac{1}{T} \int_{t_1}^{t_1+T} x(t)dt$$

and the real analysis equation gives

$$\frac{1}{2}a_0 = \frac{1}{T} \int_{t_1}^{t_1+T} x(t)dt$$

which are both simply the average value of the function over one complete period. If a function $x(t)$ is modified by adding a constant value to it, the only change in its series representation is in the coefficient of the zero-frequency term, either X_0 or a_0.

■ **Example 4**

Find a Fourier series representation for the periodic "saw-tooth" waveform with period T

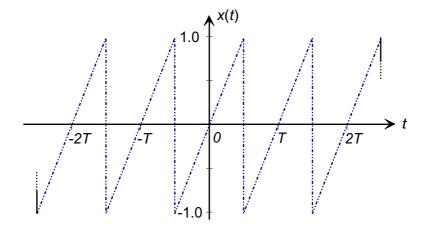$$x(t) = \frac{2}{T}t, \qquad |t| < T/2$$

shown in Fig. 5.

Figure 5: A periodic saw-tooth waveform.

Solution: The complex Fourier coefficients are

$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} \frac{2}{T} t e^{jn\Omega_0 t} dt, \tag{i}$$

and integrating by parts

$$
\begin{aligned}
X_n &= \frac{2j}{n\Omega_0 T^2} \left[te^{-jn\Omega_0 t} \Big|_{-T/2}^{T/2} + \int_{-T/2}^{T/2} \frac{1}{jn\Omega_0} e^{-jn\Omega_0 t} dt \right] \\
&= \frac{j}{2n\pi} \left[e^{-jn\pi} + e^{jn\pi} \right] + 0 \\
&= \frac{j}{n\pi} \cos(n\pi) \\
&= \frac{j(-1)^n}{n\pi} \qquad n \neq 0, \tag{ii}
\end{aligned}
$$

since $\cos(n\pi) = (-1)^n$. The zero frequency coefficient must be evaluated separately:

$$X_0 = \frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{2}{T} t \right) dt = 0. \tag{iii}$$

The Fourier series is:

$$
\begin{aligned}
x(t) &= \sum_{n=1}^{\infty} \frac{j(-1)^n}{n\pi} \left(e^{jn\Omega_0 t} - e^{-jn\Omega_0 t} \right) \\
&= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\Omega_0 t) \\
&= \frac{2}{\pi} \left(\sin(\Omega_0 t) - \frac{1}{2}\sin(2\Omega_0 t) + \frac{1}{3}\sin(3\Omega_0 t) - \frac{1}{4}\sin(4\Omega_0 t) + \dots \right). \tag{iv}
\end{aligned}
$$

■ Example 5

Find a Fourier series representation of the function

$$x(t) = \begin{cases} \dfrac{1}{4} + \dfrac{2}{T}t & -5T/8 \leq t < -T/8, \\[2ex] \dfrac{5}{4} + \dfrac{2}{T}t & -T/8 \leq t < 3T/8 \end{cases}$$

as shown in Fig. 6.
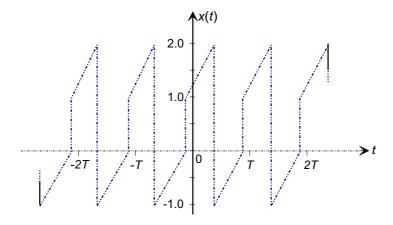
Solution: If $x(t)$ is rewritten as:

Figure 6: A modified saw-tooth function having a shift in amplitude and time origin

$$x(t) = \begin{cases} 0 + \dfrac{2}{T}(t + T/8) & -5T/8 \leq t < -T/8, \\[2ex] 1 + \dfrac{2}{T}(t + T/8) & -T/8 \leq t < 3T/8, \end{cases} \tag{i}$$

then

$$x(t) = x_1(t + T/8) + x_2(t + T/8), \tag{ii}$$

where $x_1(t)$ is the square wave function analysed in Example 3:

$$x(t) = \begin{cases} 1 & 0 \leq t < T/2, \\ 0 & T/2 \leq t < T, \end{cases} \tag{iii}$$

and $x_2(t)$ is the saw-tooth function of Example 4.

$$x(t) = \frac{2}{T}t, \qquad |t| < T/2. \tag{iv}$$

Therefore the function $x(t)$ is a *time shifted* version of the *sum* of two functions for which we already know the Fourier series. The Fourier series for $x(t)$ is the sum (Linearity Property) of phase shifted versions (Time Shifting Property) of the pair of Fourier

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series derived in Examples 3 and 4. The time shift of $T/8$ seconds adds a phase shift of $n\Omega_0 T/8 = n\pi/4$ radians to each component

$$
\begin{aligned}
x_1(t + T/8) &= \frac{1}{2} + \frac{2}{\pi}\left(\sin(\Omega_0 t + \pi/4) + \frac{1}{3}\sin(3\Omega_0 t + 3\pi/4)\right. \\
&\quad \left. + \frac{1}{5}\sin(5\Omega_0 t + 5\pi/4) + \frac{1}{7}\sin(7\Omega_0 t + 7\pi/4) + \ldots\right) \quad \text{(v)} \\
x_2(t + T/8) &= \frac{2}{\pi}\left(\sin(\Omega_0 t + \pi/4) - \frac{1}{2}\sin(2\Omega_0 t + \pi/2) + \frac{1}{3}\sin(3\Omega_0 t + 3\pi/4)\right. \\
&\quad \left. - \frac{1}{4}\sin(4\Omega_0 t + \pi) + \frac{1}{5}\sin(5\Omega_0 t + 5\pi/4) - \ldots\right). \quad \text{(vi)}
\end{aligned}
$$

The sum of these two series is

$$
\begin{aligned}
x(t) &= x_1(t) + x_2(t) \\
&= \frac{1}{2} + \frac{2}{\pi}\left(2\sin(\Omega_0 t + \pi/4) - \frac{1}{2}\sin(2\Omega_0 t + \pi/2) + \frac{2}{3}\sin(3\Omega_0 t + 3\pi/4)\right. \quad \text{(vii)} \\
&\quad \left. - \frac{1}{4}\sin(4\Omega_0 t + \pi) + \frac{2}{5}\sin(5\Omega_0 t + 5\pi/4) - \frac{1}{6}\sin(6\Omega_0 t + 3\pi/2) + \ldots\right). \quad \text{(viii)}
\end{aligned}
$$

---

3   The Response of Linear Systems to Periodic Inputs

Consider a linear single-input, single-output system with a frequency response function $H(j\Omega)$. Let the input $u(t)$ be a periodic function with period T, and assume that all initial condition transient components in the output have decayed to zero. Because the input is a periodic function it can be written in terms of its complex or real Fourier series

$$
u(t) = \sum_{n=-\infty}^{\infty} U_n e^{jn\Omega_0 t} \tag{23}
$$

$$
= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \mathcal{A}_n \sin(n\Omega_0 t + \phi_n) \tag{24}
$$

The nth real harmonic input component, $u_n(t) = \mathcal{A}_n \sin(n\Omega_0 t + \phi_n)$, generates an output sinusoidal component $y_n(t)$ with a magnitude and a phase that is determined by the system's frequency response function $H(j\Omega)$:

$$
y_n(t) = |H(jn\Omega_0)| \, \mathcal{A}_n \sin(n\Omega_0 t + \phi_n + \angle H(jn\Omega_0)). \tag{25}
$$

The principle of superposition states that the total output $y(t)$ is the sum of all such component outputs, or

$$
\begin{aligned}
y(t) &= \sum_{n=0}^{\infty} y_n(t) \\
&= \frac{1}{2}a_0 H(j0) + \sum_{n=1}^{\infty} \mathcal{A}_n \, |H(jn\Omega_0)| \sin\left(n\Omega_0 t + \phi_n + \angle H(jn\Omega_0)\right), \tag{26}
\end{aligned}
$$

which is itself a Fourier series with the same fundamental and harmonic frequencies as the input. The output $y(t)$ is therefore also a periodic function with the same period T as the input, but

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because the system frequency response function has modified the relative magnitudes and the phases of the components, the waveform of the output $y(t)$ differs in form and appearance from the input $u(t)$.

In the complex formulation the input waveform is decomposed into a set of complex exponentials $u_n(t) = U_n e^{jn\Omega_0 t}$. Each such component is modified by the system frequency response so that the output component is

$$y_n(t) = H(jn\Omega_0)U_n e^{jn\Omega_0 t} \qquad (27)$$

and the complete output Fourier series is

$$y(t) = \sum_{n=-\infty}^{\infty} y_n(t) = \sum_{n=-\infty}^{\infty} H(jn\Omega_0)U_n e^{jn\Omega_0 t}. \qquad (28)$$

■ Example 6

The first order electrical network shown in Fig. 7 is excited with the saw-tooth function discussed in Example 4. Find an expression for the series representing the output $V_o(t)$.
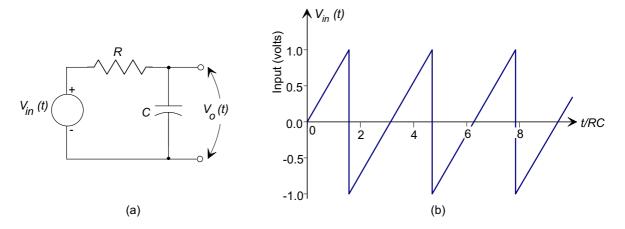

Figure 7: A first-order electrical system (a), driven by a saw-tooth input waveform (b).

Solution: The electrical network has a transfer function

$$H(s) = \frac{1}{RCs + 1}, \qquad (i)$$

and therefore has a frequency response function

$$|H(j\Omega)| = \frac{1}{\sqrt{(\Omega RC)^2 + 1}} \qquad (ii)$$

$$\angle H(j\Omega) = \tan^{-1}(-\Omega RC). \qquad (iii)$$

From Example 4, the input function $u(t)$ may be represented by the Fourier series

$$u(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\Omega_0 t). \qquad (iv)$$

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At the output the series representation is

$$
\begin{aligned}
y(t) &= \sum_{n=1}^{\infty} |H(jn\Omega_0)| \frac{2(-1)^{n+1}}{n\pi} \sin\left(n\Omega_0 t + \angle H(jn\Omega_0)\right) \\
&= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi\sqrt{(n\Omega_0 RC)^2 + 1}} \sin\left(n\Omega_0 t + \tan^{-1}(-n\Omega_0 RC)\right). \quad\quad \text{(v)}
\end{aligned}
$$

As an example consider the response if the period of the input is chosen to be $T = \pi RC$, so that $\Omega_0 = 2/(RC)$, then

$$
y(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi\sqrt{(2n)^2 + 1}} \sin\left(\frac{2n}{RC}t + \tan^{-1}(-2n)\right).
$$

Figure 8 shows the computed response, found by summing the first 100 terms in the Fourier Series.
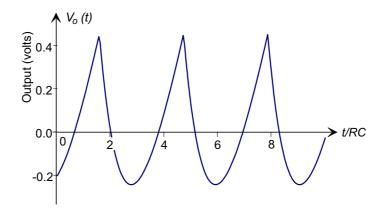

Figure 8: Response of first-order electrical system to a saw-tooth input

Equations (28) and (27) show that the output component Fourier coefficients are products of the input component coefficient and the frequency response evaluated at the frequency of the harmonic component. No new frequency components are introduced into the output, but the form of the output $y(t)$ is modified by the redistribution of the input component amplitudes and phase angles by the frequency response $H(jn\Omega_0)$. If the system frequency response exhibits a low-pass characteristic with a cut-off frequency within the spectrum of the input $u(t)$, the high frequency components are attenuated in the output, with a resultant general "rounding" of any discontinuities in the input. Similarly a system with a high-pass characteristic emphasizes any high frequency component in the input. A system having lightly damped complex conjugate pole pairs exhibits resonance in its response at frequencies close to the undamped natural frequency of the pole pair. It is entirely possible for a periodic function to excite this resonance through one of its harmonics even though the fundamental frequency is well removed from the resonant frequency as is shown in Example 7.

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■ Example 7

A cart, shown in Fig. 9a, with mass $m = 1.0$ kg is supported on low friction bearings that exhibit a viscous drag $B = 0.2$ N-s/m, and is coupled through a spring with stiffness K $= 25$ N/m to a velocity source with a magnitude of 10 m/s, but which switches direction every π seconds as shown in Fig. 9b.

$$V_{in}(t) = \begin{cases} 10 \text{ m/sec} & 0 \le t < \pi, \\ -10 \text{ m/sec} & \pi \le t < 2\pi \end{cases}$$

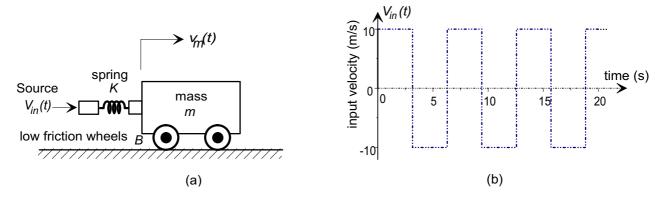The task is to find the resulting velocity of the mass $v_m(t)$.

Figure 9: A second-order system and its linear graph, together with its input waveform $V_{in}(t)$.

Solution: The system has a transfer function

$$H(s) \quad = \quad \frac{K/m}{s^2 + (B/m)\,s + K/m} \tag{i}$$

$$= \quad \frac{25}{s^2 + 0.2s + 25} \tag{ii}$$

and an undamped natural frequency $\Omega_n = 5$ rad/s and a damping ratio $\zeta = 0.02$. It is therefore lightly damped and has a strong resonance in the vicinity of 5 rad/s.

The input $\Omega(t)$ has a period of $T = 2\pi$ s, and fundamental frequency of $\Omega_0 = 2\pi/T = 1$ rad/s. The Fourier series for the input may be written directly from Example 3, and contains only odd harmonics:

$$u(t) \quad = \quad \frac{20}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left((2n-1)\Omega_0 t\right) \tag{iii}$$

$$= \quad \frac{20}{\pi} \left(\sin(\Omega_0 t) + \frac{1}{3}\sin(3\Omega_0 t) + \frac{1}{5}\sin(5\Omega_0 t) + \dots \right). \tag{iv}$$

From Eq. (ii) the frequency response of the system is

$$H(j\Omega) = \frac{25}{(25 - \Omega^2) + j0.2\Omega}, \tag{v}$$

which when evaluated at the harmonic frequencies of the input $n\Omega_0 = n$ radians/sec. is

$$H(jn\Omega_0) = \frac{25}{(25 - n^2) + j0.2n}. \tag{vi}$$

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The following table summarizes the the first five odd spectral components at the system input and output. Fig. 10a shows the computed frequency response magnitude for the system and the relative gains and phase shifts (rad) associated with the first five terms in the series.

$n\Omega_0$	u_n	$|H(jn\Omega_0)|$	$\angle H(jn\Omega_0)$	y_n
1	$6.366\sin(t)$	1.041	-0.008	$6.631\sin(t-0.008)$
3	$2.122\sin(3t)$	1.561	-0.038	$3.313\sin(t-0.038)$
5	$1.273\sin(5t)$	25.00	-1.571	$31.83\sin(t-1.571)$
7	$0.909\sin(7t)$	1.039	-3.083	$0.945\sin(t-3.083)$
9	$0.707\sin(9t)$	0.446	-3.109	$0.315\sin(t-3.109)$

The resonance in $|H(j\Omega)|$ at the undamped natural frequency $\Omega_n = 5$ rad/s has a large effect on the relative amplitude of the 5th harmonic in the output $y(t)$. Figure 10b shows the system input and output waveforms. The effect of the resonance can be clearly seen, for the output appears to be almost sinusoidal at a frequency of 5 rad/s. In fact the output is still a periodic waveform with a period of 2π seconds but the fifth harmonic component dominates the response and makes it appear to be sinusoidal at its own frequency.
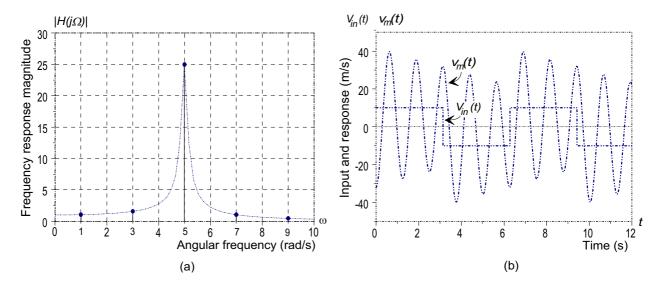

Figure 10: (a) The frequency response magnitude function of the mechanical system in Example 8, and (b) an input square wave function and its response.

4   Fourier Analysis of Transient Waveforms

Many waveforms found in practice are not periodic and therefore cannot be analyzed directly using Fourier series methods. A large class of system excitation functions can be characterized as *aperiodic*, or transient, in nature. These functions are limited in time, they occur only once, and decay to zero as time becomes large [2,4,5].
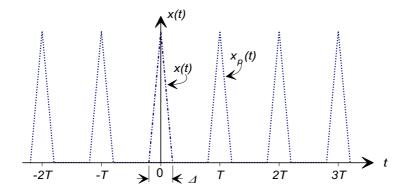
Figure 11: Periodic extension of a transient waveform

Consider a function $x(t)$ of duration Δ that exists only within a defined interval $t_1 < t \leq t_1 + \Delta$, and is identically zero outside of this interval. We begin by making a simple assumption; namely that in observing the transient phenomenon $x(t)$ within any finite interval that encompasses it, we have observed a fraction of a single period of a periodic function with a very large period; much larger than the observation interval. Although we do not know what the duration of this hypothetical period is, it is assumed that $x(t)$ will repeat itself at some time in the distant future, but in the meantime it is assumed that this periodic function remains identically zero for the rest of its period outside the observation interval.

The analysis thus conjectures a new function $x_p(t)$, known as a *periodic extension* of $x(t)$, that repeats every T seconds $(T > \Delta)$, but at our discretion we can let T become very large. Figure 11 shows the hypothetical periodic extension $x_p(t)$ created from the observed $x(t)$. As observers of the function $x_p(t)$ we need not be concerned with its pseudo-periodicity because we will never be given the opportunity to experience it outside the first period, and furthermore we can assume that if $x_p(t)$ is the input to a linear system, T is so large that the system response decays to zero before the arrival of the second period. Therefore we assume that the response of the system to $x(t)$ and $x_p(t)$ is identical within our chosen observation interval. The important difference between the two functions is that $x_p(t)$ is periodic, and therefore has a Fourier series description.

The development of Fourier analysis methods for transient phenomena is based on the limiting behavior of the Fourier series describing $x_p(t)$ as the period T approaches infinity. Consider the behavior of the Fourier series of a simple periodic function as its period T is varied; for example an even periodic pulse function $x(t)$ of fixed width Δ:

$$x(t) = \begin{cases} 1 & |t| < \Delta/2 \\ 0 & \Delta/2 \leq |t| \leq T - \Delta/2 \end{cases} \qquad (29)$$

as shown in Figure 12. Assume that the pulse width Δ remains constant as the period T varies. The Fourier coefficients in complex form are

$$\begin{aligned} X_n &= \frac{1}{T} \int_{-\Delta/2}^{\Delta/2} e^{-jn\Omega_0 t} dt \\ &= \frac{j}{2n\pi} \left[e^{-jn\Omega_0 \Delta/2} - e^{jn\Omega_0 \Delta/2} \right] \\ &= \frac{1}{n\pi} \sin\left(n\pi\Delta/T\right) \\ &= \frac{\Delta}{T} \frac{\sin\left(n\pi\Delta/T\right)}{\left(n\pi\Delta/T\right)} \qquad n \neq 0 \end{aligned} \qquad (30)$$
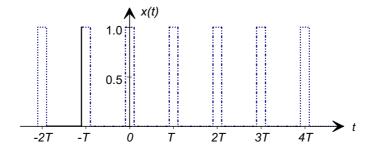
19

Figure 12: A periodic rectangular pulse function of fixed duration Δ but varying period T.

and

$$X_0 = \frac{1}{T} \int_{-\Delta/2}^{\Delta/2} 1 \, dt = \frac{\Delta}{T} \tag{31}$$

and the spectral lines are spaced along the frequency axis at intervals of $\Omega_0 = 2\pi/T$ rad/s. To investigate the behavior of the spectrum as the period T is altered, we define a continuous function of frequency

$$X(\Omega) = \frac{\sin(\Omega\Delta/2)}{(\Omega\Delta/2)}$$

and note that the Fourier coefficients may be computed directly from $X(\Omega)$

$$X_n = \left. \frac{\Delta}{T} \frac{\sin(\Omega\Delta/2)}{(\Omega\Delta/2)} \right|_{\Omega=2\pi n/T} \tag{32}$$

$$= \left. \frac{\Delta}{T} X(\Omega) \right|_{\Omega=n\Omega_0} \tag{33}$$

The function $X(\Omega)$ depends only on the pulse width Δ, and is independent of the period T. As T is changed, apart from the amplitude scaling factor Δ/T, the frequency dependence of the Fourier coefficients is defined by $X(\Omega)$; the relative strength of the nth complex harmonic component, at a frequency $n\Omega_0$, is defined by $X(n\Omega_0)$. The function $X(\Omega)$ is therefore an *envelope* function that depends only on $x(t)$ and not on the length of the assumed period. Figure 13 shows examples of the line spectra for the periodic pulse train as the period T is changed. The following general observations on the behavior of the Fourier coefficients as T varies can be made:

(1) As the repetition period T increases, the fundamental frequency Ω_0 decreases, and the spacing between adjacent lines in the spectrum decreases.

(2) As the repetition period T increases, the scaling factor Δ/T decreases, causing the magnitude of all of the spectral lines to be diminished. In the limit as T approaches infinity, the amplitude of the individual lines becomes infinitesimal.

(3) The "shape" of the spectrum is defined by the function $X(\Omega)$ and is independent of T.

Assume that we have an aperiodic function $x(t)$ that is non-zero only for a defined time interval Δ, and without loss of generality assume that the interval is centered around the time origin ($t = 0$). Then assume a periodic extension $x_p(t)$ of $x(t)$ with period T that fully encompasses the interval Δ. The Fourier series description of $x_p(t)$ is contained in the analysis and synthesis equations

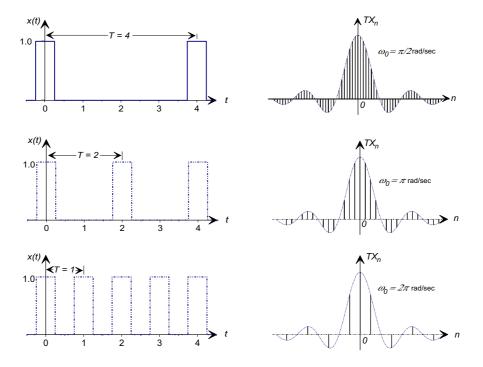$$X_n = \frac{1}{T} \int_{-T/2}^{T/2} x_p(t) e^{-jn\Omega_0 t} dt \tag{34}$$

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Figure 13: Line spectra of periodic extensions of an even rectangular pulse function.

$$x_p(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t}. \tag{35}$$

These two equations may be combined by substituting for X_n in the synthesis equation,

$$x_p(t) = \sum_{n=-\infty}^{\infty} \left\{ \frac{\Omega_0}{2\pi} \int_{-T/2}^{T/2} x_p(t) e^{-jn\Omega_0 t} dt \right\} e^{jn\Omega_0 t} \tag{36}$$

where the substitution $\Omega_0/2\pi = 1/T$ has also been made.

The period T is now allowed to become arbitrarily large, with the result that the fundamental frequency Ω_0 becomes very small and we write $\Omega_0 = \delta\Omega$. We define $x(t)$ as the limiting case of $x_p(t)$ as T approaches infinity, that is

$$
\begin{aligned}
x(t) &= \lim_{T \to \infty} x_p(t) \\
&= \lim_{T \to \infty} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \left\{ \int_{-T/2}^{T/2} x_p(t) e^{-jn\delta\Omega t} dt \right\} e^{jn\delta\Omega t} \delta\Omega \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \left\{ \int_{-\infty}^{-\infty} x(t) e^{-j\Omega t} dt \right\} e^{j\Omega t} d\Omega
\end{aligned}
\tag{37}
$$

where in the limit the summation has been replaced by an integral. If the function inside the braces is defined to be $X(j\Omega)$, Equation 37 may be expanded into a pair of equations, known as the *Fourier transform pair*:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \tag{38}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \tag{39}$$

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which are the equations we seek.

Equation 38 is known as the *forward* Fourier transform, and is analogous to the analysis equation of the Fourier series representation. It expresses the time-domain function $x(t)$ as a function of frequency, but unlike the Fourier series representation it is a continuous function of frequency. Whereas the Fourier series coefficients have units of amplitude, for example volts or Newtons, the function $X(j\Omega)$ has units of amplitude density, that is the total "amplitude" contained within a small increment of frequency is $X(j\Omega)\delta\Omega/2\pi$.

Equation 39 defines the *inverse* Fourier transform. It allows the computation of the time-domain function from the frequency domain representation $X(j\Omega)$, and is therefore analogous to the Fourier series synthesis equation. Each of the two functions $x(t)$ or $X(j\Omega)$ is a complete description of the function and Equations 38 and 39 allow the transformation between the domains.

We adopt the convention of using lower-case letters to designate time-domain functions, and the same upper-case letter to designate the frequency-domain function. We also adopt the nomenclature

$$x(t) \overset{\mathcal{F}}{\Longleftrightarrow} X(j\Omega)$$

as denoting the bidirectional Fourier transform relationship between the time and frequency-domain representations, and we also frequently write

$$
\begin{aligned}
X(j\Omega) &= \mathcal{F}\{x(t)\} \\
x(t) &= \mathcal{F}^{-1}\{X(j\Omega)\}
\end{aligned}
$$

as denoting the operation of taking the forward $\mathcal{F}\{\}$, and inverse $\mathcal{F}^{-1}\{\}$ Fourier transforms respectively.

4.1 Alternate Definitions

Although the definitions of Eqs. (38) and (39) flow directly from the Fourier series, definitions for the Fourier transform varies appreciably from text to text and in different disciplines. The main objection to the convention adopted here is the asymmetry introduced by the factor $1/2\pi$ that appears in the inverse transform. Some authors, usually in physics texts, define the so-called *unitary* Fourier transform pair as

$$
\begin{aligned}
X(j\Omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \\
x(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t}d\Omega
\end{aligned}
$$

so as to distribute the constant symmetrically over the forward and inverse transforms.

Many engineering texts address the issue of the asymmetry by defining the transform with respect to frequency $F = 2\pi\Omega$ in Hz, instead of angular frequency Ω in radians/s. The effect, through the change in the variable in the inverse transform, is to redefine the transform pair as

$$
\begin{aligned}
X(F) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi F t}dt \\
x(t) &= \int_{-\infty}^{\infty} X(F)e^{j2\pi F t}dF
\end{aligned}
$$

Some authors also adopt the notation of dropping the j from the frequency domain representation and write $X(\Omega)$ or $X(F)$ as above.

Even more confusing is the fact that some authors (particularly in physics) adopt a definition that reverses the sign convention on the exponential terms in the Fourier integral, that is they define

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{j\Omega t}dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{-j\Omega t}d\Omega$$

These various definitions of the transform pair mean that care must be taken to understand the particular definition adopted by texts and software packages. Throughout this course we will retain the definitions in Eqs. (38) and (39).

4.2   Fourier Transform Examples

In this section we present five illustrative examples of Fourier transforms of common time domain functions.

■ Example 8

Find the Fourier transform of the pulse function

$$x(t) = \begin{cases} a & |t| < T/2 \\ 0 & \text{otherwise.} \end{cases}$$
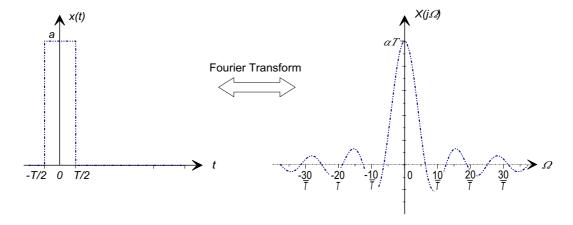
shown in Figure 14.

Figure 14: An even aperiodic pulse function and its Fourier transform

Solution: From the definition of the forward Fourier transform

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{i}$$

$$= a\int_{-T/2}^{T/2} e^{-j\Omega t}dt \tag{ii}$$

$$= a\left[\frac{j}{\Omega}e^{-j\Omega t}\Big|_{-T/2}^{T/2}\right] \tag{iii}$$

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$$= \frac{ja}{\Omega} \left[e^{-j\Omega T/2} - e^{j\Omega T/2} \right] \tag{iv}$$

$$= aT \frac{\sin(\Omega T/2)}{\Omega T/2}. \tag{v}$$

The Fourier transform of the rectangular pulse is a real function, of the form $(\sin x)/x$ centered around the $j\Omega = 0$ axis. Because the function is real, it is sufficient to plot a single graph showing only $|X(j\Omega)|$ as in Figure 14. Notice that while $X(j\Omega)$ is a generally decreasing function of Ω it never becomes identically zero, indicating that the rectangular pulse function contains frequency components at all frequencies.

The function $(\sin x)/x = 0$ when the argument $x = n\pi$ for any integer n ($n \neq 0$). The main peak or "lobe" of the spectrum $X(j\Omega)$ is therefore contained within the frequency band defined by the first two zero-crossings $|\Omega T/2| < \pi$ or $|\Omega| < 2\pi/T$. Thus as the pulse duration T is decreased, the spectral bandwidth of the pulse increases as shown in Figure 15, indicating that short duration pulses have a relatively larger high frequency content.
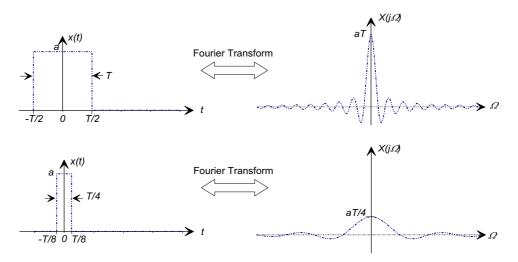

Figure 15: Dependence of the bandwidth of a pulse on its duration

■ Example 9

Find the Fourier transform of the Dirac delta function $\delta(t)$.

Solution: The delta impulse function, is an important theoretical function in system dynamics, defined as

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0, \end{cases} \tag{i}$$

with the additional defining property that

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

The delta function exhibits a "sifting" property when included in an integrand:

$$\int_{-\infty}^{\infty} \delta(t-T)x(t)dt = x(T). \tag{ii}$$

When substituted into the forward Fourier transform

$$\begin{aligned}\Delta(j\Omega) &= \int_{-\infty}^{\infty} \delta(t)e^{-j\Omega t}dt \\ &= 1\end{aligned} \tag{iii}$$

by the sifting property. The spectrum of the delta function is therefore constant over all frequencies. It is this property that makes the impulse a very useful test input for linear systems.

■ Example 10

Find the Fourier transform of a finite duration sinusoidal "tone-burst"

$$x(t) = \begin{cases} \sin \Omega_0 t & |t| < T/2 \\ 0 & \text{otherwise.} \end{cases}$$
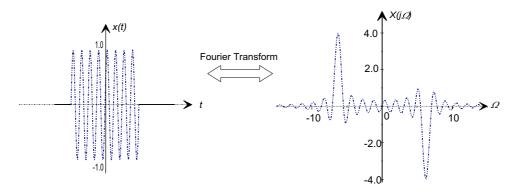
shown in Figure 16.

Figure 16: A sinusoidal tone-burst and its Fourier transform

Solution: The forward Fourier transform is

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{i}$$

$$= \int_{-T/2}^{T/2} \sin(\Omega_0 t)e^{-j\Omega t}dt. \tag{ii}$$

The sinusoid is expanded using the Euler formula:

$$X(j\Omega) = \frac{1}{j2}\int_{-T/2}^{T/2} \left[e^{-j(\Omega-\Omega_0)t} - e^{-j(\Omega+\Omega_0)t}\right]dt \tag{iii}$$

$$= \frac{1}{2(\Omega-\Omega_0)}\left[e^{-j(\Omega-\Omega_0)t}\Big|_{-T/2}^{T/2} - \frac{1}{2(\Omega+\Omega_0)}\left[e^{-j(\Omega+\Omega_0)t}\Big|_{-T/2}^{T/2}\right. \tag{iv}$$

$$= -j\frac{T}{2}\left\{\frac{\sin\left((\Omega-\Omega_0)T/2\right)}{(\Omega-\Omega_0)T/2} - \frac{\sin\left((\Omega+\Omega_0)T/2\right)}{(\Omega+\Omega_0)T/2}\right\} \tag{v}$$

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which is a purely imaginary odd function that is the sum of a pair of shifted imaginary $(\sin x)/x$ functions, centered on frequencies $\pm\Omega_0$, as shown in Fig. 16. The zero-crossings of the main lobe of each function are at $\Omega_0 \pm 2\pi/T$ indicating again that as the duration of a transient waveform is decreased its spectral width increases. In this case notice that as the duration T is increased the spectrum becomes narrower, and in the limit as $T \to \infty$ it achieves zero width and becomes a simple line spectrum.

■ Example 11

Find the Fourier transform of the one-sided real exponential function

$$x(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0. \end{cases}$$

(for $a > 0$) as shown in Figure 17.

Figure 17: The one-sided real exponential function and its spectrum

Solution: From the definition of the forward Fourier transform

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt & \text{(i)} \\ &= \int_{0}^{\infty} e^{-at}e^{-j\Omega t}dt & \text{(ii)} \\ &= \left[\frac{-1}{a+j\Omega}e^{-(a+j\Omega)t}\right]_{0}^{\infty} & \text{(iii)} \\ &= \frac{1}{a+j\Omega} & \text{(iv)} \end{aligned}$$

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which is complex, and in terms of a magnitude and phase function is

$$|X(j\Omega)| = \frac{1}{\sqrt{a^2 + \Omega^2}} \tag{v}$$

$$\angle X(j\Omega) = \tan^{-1}\left(\frac{-\Omega}{a}\right) \tag{vi}$$

■ Example 12

Find the Fourier transform of a damped one-sided sinusoidal function

$$x(t) = \begin{cases} 0 & t < 0 \\ e^{-\sigma t}\sin\Omega_0 t & t \geq 0. \end{cases}$$
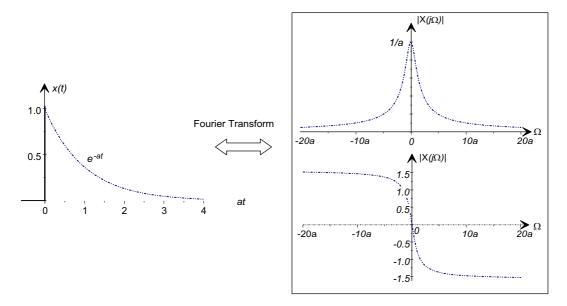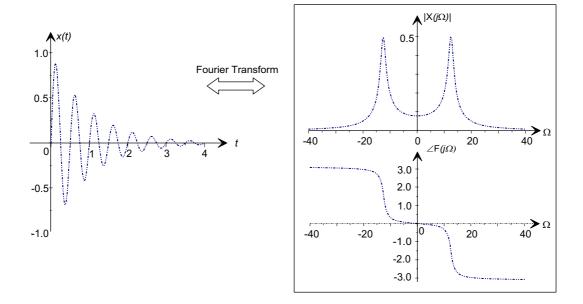
(for $\sigma > 0$) as shown in Figure 18.

Figure 18: A damped sinusoidal function and its spectrum

Solution: From the definition of the forward Fourier transform

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{i}$$

$$= \frac{1}{j2}\int_{0}^{\infty} e^{-\sigma t}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)e^{-j\Omega t}dt \tag{ii}$$

$$= \frac{1}{j2}\left[-\frac{1}{\sigma + j(\Omega_0 - \Omega)}e^{-(\sigma+j(\Omega-\Omega_0))t}\Big|_{0}^{\infty} - \tag{iii}\right.$$

$$\frac{1}{j2}\left[-\frac{1}{\sigma + j(\Omega_0 + \Omega)}e^{-(\sigma+j(\Omega+\Omega_0))t}\Big|_{0}^{\infty}\right. \tag{iv}$$

$$= \frac{\Omega_0}{(\sigma + j\Omega)^2 + \Omega_0^2}. \tag{v}$$

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The magnitude and phase of this complex quantity are

$$|X(j\Omega)| = \frac{\Omega_0}{\sqrt{(\sigma^2 + \Omega_0^2 - \Omega^2)^2 + (2\sigma\Omega)^2}} \qquad \text{(vi)}$$

$$\angle X(j\Omega) = \tan^{-1} \frac{-2\sigma\Omega}{\sigma^2 + \Omega_0^2 - \Omega^2} \qquad \text{(vii)}$$

4.3   Properties of the Fourier Transform

A full description of the properties of the Fourier transform is beyond the scope and intent of this handout, and the interested reader is referred to the many texts devoted to the Fourier transform [2,4,5]. The properties listed below are presented because of their importance in the study of system dynamics. In the following properties it is assumed that

$$x(t) \overset{\mathcal{F}}{\Longleftrightarrow} X(j\Omega)$$

(1) **Existence of the Fourier Transform**   A modified form of the three Dirichlet conditions presented for the Fourier series guarantees the existence of the Fourier transform. These are sufficient conditions, but are not strictly necessary. For the Fourier transform the conditions are

- The function x(t) must be *integrable in the absolute sense* over all time, that is
$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty.$$

- There must be at most a finite number of maxima and minima in the function $x(t)$. Notice that periodic functions are excluded by this and the previous condition.

- There must be at most a finite number of discontinuities in the function $x(t)$, and all such discontinuities must be finite in magnitude.

(2) **Linearity of the Fourier Transform**   Like the Fourier series, the Fourier transform is a linear operation. If

$$g(t) \overset{\mathcal{F}}{\Longleftrightarrow} G(j\Omega) \quad \text{and} \quad h(t) \overset{\mathcal{F}}{\Longleftrightarrow} H(j\Omega)$$

then for arbitrary constants a and b,

$$ag(t) + bh(t) \overset{\mathcal{F}}{\Longleftrightarrow} aG(j\Omega) + bH(j\Omega). \qquad (40)$$

(3) **Duality**   If

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t} d\Omega \overset{\mathcal{F}}{\Longleftrightarrow} X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt$$

then by definition

$$\mathcal{F}\{X(jt)\} = \int_{-\infty}^{\infty} X(jt)e^{-j\Omega t} dt = \int_{-\infty}^{\infty} X(jt)e^{j(-\Omega)t} dt = 2\pi x(-\Omega)$$

or

$$X(jt) \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi x(-\Omega)$$

where $X(jt)$ is $X(j\Omega)$ where Ω has been replaced by t, and $x(-\Omega)$ is $x(t)$ where t is replaced by $-\Omega$. Therefore if we know the Fourier transform of one function, we also know it for another.

(4) Even and Odd Functions Let $x(t)$ be a real function, and write it in terms of an even function $x_e(t)$ and an odd function $x_o(t)$

$$x(t) = x_e(t) + x_o(t).$$

Then the Fourier transform is

$$
\begin{aligned}
X(j\Omega) &= \int_{-\infty}^{\infty} (x_e(t) + xc_o(t)) \left(\cos(\Omega t) - j\sin(\Omega t) \right) dt \\
&= \int_{-\infty}^{\infty} x_e(t) \cos(\Omega t) dt - j \int_{-\infty}^{\infty} x_o(t) \sin(\Omega t) dt \quad\quad (41)
\end{aligned}
$$

since odd terms in the integrand contribute zero to the integral. Equation (41) allows the following conclusions:

- For real $x(t)$ the Fourier transform shows conjugate symmetry, that is

$$X(-j\Omega) = \overline{X(j\Omega)}. \quad\quad (42)$$

- The Fourier transform of a real even function of time is a real even function
- The Fourier transform of an real odd function is an imaginary odd function.

Similar arguments may be used to make similar conclusions for the inverse Fourier transform, for example if $X(j\Omega)$ is a real function, then

$$x(-t) = \overline{x(t)}.$$

(5) Time Shifting The Fourier transform of $x(t + \tau)$, a time shifted version of $x(t)$, is

$$\mathcal{F}\{x(t+\tau)\} = e^{j\Omega\tau} X(j\Omega).$$

This result can be shown easily, since by definition

$$\mathcal{F}\{x(t+\tau)\} = \int_{-\infty}^{\infty} x(t+\tau) e^{-j\Omega t} dt.$$

With a change of variable $\nu = t + \tau$,

$$
\begin{aligned}
\mathcal{F}\{x(t+\tau)\} &= \int_{-\infty}^{\infty} x(\nu) e^{-j\Omega(\nu-\tau)} d\nu \\
&= e^{j\Omega\tau} \int_{-\infty}^{\infty} x(\nu) e^{-j\Omega\nu} d\nu \\
&= e^{j\Omega\tau} X(j\Omega) \\
&= |X(j\Omega)| e^{j(\angle X(j\Omega) + \Omega\tau)}. \quad\quad (43)
\end{aligned}
$$

which indicates that the Fourier transform of a time shifted waveform has the same magnitude function as the original waveform, but has an additional phase-shift term that is directly proportional to frequency.

(4) Time Scaling

$$x(at) \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{|a|} X\left(j\Omega/a\right), \qquad a \neq 0 \tag{44}$$

(9) Time Reversal   If $a = -1$, the time scaling property gives

$$x(-t) \overset{\mathcal{F}}{\Longleftrightarrow} X(-j\Omega). \tag{45}$$

(5) Waveform Energy   We have asserted that the time domain representation $x(t)$ and the frequency domain representation $X(j\Omega)$ are both complete descriptions of the function related through the Fourier transform

$$x(t) \overset{\mathcal{F}}{\Longleftrightarrow} X(j\Omega).$$

If we consider the function $x(t)$ to be a system variable, for example a voltage or current in an electrical system, the instantaneous power dissipated in a resistive element with a value of unity is equal to $x^2(t)$. For example, the power dissipated when voltage $v(t)$ is applied to an electrical resistance of 1 ohm is $v^2(t)$. The power associated with a complex variable $v(t)$ is $|v(t)|^2$. The "energy" of an aperiodic function in the time domain is defined as the integral of this hypothetical instantaneous power over all time

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{46}$$

Parseval's theorem [3] asserts the equivalence of the total waveform energy in the time and frequency domains by the relationship

$$
\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 \, dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 \, d\Omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)\overline{X(j\Omega)} d\Omega.
\end{aligned}
\tag{47}
$$

In other words, the quantity $|X(j\Omega)|^2$ is a measure of the energy of the function per unit bandwidth.

The energy ΔE contained between two angular frequencies Ω_1 and Ω_2 is

$$\Delta E = \frac{1}{2\pi} \int_{\Omega_1}^{\Omega_2} X(j\Omega)\overline{X(j\Omega)} d\Omega. \tag{48}$$

Notice that this is a one-sided energy content and that because the Fourier transform is a two-sided spectral representation, the total energy in a real function includes contributions from both positive and negative frequencies. The function

$$\Phi(j\Omega) = |X(j\Omega)|^2$$

is a very important quantity in experimental system dynamics, and is known as the *energy density* spectrum. It is a real function, with units of energy per unit bandwidth, and shows how the energy of a waveform $x(t)$ is distributed across the spectrum.

**(6) Relationship Between the Fourier Transform and the Fourier
Series of a Periodic Extension of an Aperiodic Function**   Let $x(t)$ be a
function, with Fourier transform $X(j\Omega)$, that exists only in a defined interval
$|t| < \Delta/2$ centered on the time origin. Then if $x_p(t)$ is a periodic extension of $x(t)$,
formed by repeating $x(t)$ at intervals $T > \Delta$, each period contains $x(t)$. Then the
Fourier coefficients describing $x_p(t)$ are

$$
\begin{aligned}
c_n &= \frac{1}{T}\int_{-T/2}^{T/2} x_p(t)e^{-jn\Omega_0 t}dt \\
&= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jn\Omega_0 t}dt \\
&= \frac{1}{T}X(jn\Omega_0).
\end{aligned}
\tag{49}
$$

The Fourier coefficients of a periodic function are scaled samples of the Fourier
transform of the function contained within a single period. The transform thus
forms the envelope function for the definition of the Fourier series as discussed in
Section 2.

(7) The Fourier Transform of the Derivative of a Function   If a function $x(t)$
has a Fourier transform $X(j\Omega)$ then

$$
\mathcal{F}\left\{\frac{dx}{dt}\right\} = j\Omega X(j\Omega),
$$

which is easily shown using integration by parts:

$$
\begin{aligned}
\mathcal{F}\left\{\frac{dx}{dt}\right\} &= \int_{-\infty}^{\infty} \frac{dx}{dt}e^{-j\Omega t}dt \\
&= \left|x(t)e^{-j\Omega t}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(t)(-j\Omega)e^{-j\Omega t}dt \\
&= 0 + j\Omega X(j\Omega)
\end{aligned}
$$

since by definition $x(t) = 0$ at $t = \pm\infty$.

This result can be applied repetitively to show that the Fourier transform of the
nth derivative of $x(t)$ is

$$
\mathcal{F}\left\{\frac{d^n x}{dt^n}\right\} = (j\Omega)^n X(j\Omega)
\tag{50}
$$

(8) The Fourier Transform of the Integral of a Function   If a function $x(t)$ has
a Fourier transform $X(j\Omega)$ then

$$
\mathcal{F}\left\{\int_{-\infty}^{t} x(\nu)d\nu\right\} = \pi X(0)\delta(j\Omega) + \frac{1}{j\Omega}X(j\Omega)
\tag{51}
$$

The additive term $\pi X(j0)\delta(j\Omega)$ accounts for the possibility of a non-zero dc com-
ponent in $x(t)$, where

$$
X(j0) = \int_{-\infty}^{\infty} x(t)dt
$$

is the average value of $x(t)$.

(9) Time Reversal   If a function $x(t)$ has a Fourier transform $X(j\Omega)$ then

$$
\mathcal{F}\left\{x(-t)\right\} = X(-j\Omega).
\tag{52}
$$

4.4 Extension of the Fourier Transform to Functions for which the Fourier Integral does not Converge.

The Dirichelet conditions (described in Sec. 4.3) are *sufficient* but not *necessary* conditions for the existence of the Fourier transform. If the use of the Dirac delta function $\delta(x)$ is allowed, the Fourier transform of many functions with a non-convergent Fourier integral may de defined. In this section we examine the approach through several examples.

■ Example 13

Define the Fourier transform of a constant $x(t) = a$. The Fourier integral

$$X(j\Omega) = \int_{-\infty}^{\infty} ae^{-j\Omega t}dt$$

does not converge. In Ex. 8 the Fourier transform of a even pulse of duration T

$$g(t) = \begin{cases} a & |t| < T/2 \\ 0 & \text{otherwise.} \end{cases}$$

was shown to be

$$G(j\Omega) = aT\frac{\sin(\Omega T/2)}{\Omega T/2}.$$

It is also shown in the example that as the duration T is increased, the peak amplitude of $G(j\Omega)$ (at $j\Omega = 0$) increases, and its bandwidth decreases. Since $x(t) = a = \lim_{T\to\infty} g(t)$, we have

$$X(j\Omega) = \lim_{T\to\infty} G(j\Omega)] = \lim_{T\to\infty} \left\{ aT\frac{\sin(\Omega T/2)}{\Omega T/2} \right\} = 2\pi a\delta(\Omega)$$

where $\delta()$ is the Dirac delta function. In other words the Fourier transform of a constant is an impulse at $\Omega = 0$.

■ Example 14

Define the Fourier transform of the unit-step (Heaviside) function $u_s(t)$, where

$$u_s(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$

Clearly the Fourier integral

$$U_s(j\Omega) = \int_{\infty}^{\infty} u_s(t)e^{j\Omega t}dt = \int_{0}^{\infty} e^{-j\Omega t}dt \tag{i}$$

does not converge, and we seek an alternative approach.

Consider the one-sided real exponential function

$$x(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0. \end{cases}$$

as described in Example 12, and note that $u_s(t) = \lim_{a \to 0} x(t)$ which implies $U_s(j\Omega) = \lim_{a \to 0} X(j\Omega)$. From Example 12

$$X(j\Omega) = \frac{1}{a + j\Omega} = \frac{a}{a^2 + \Omega^2} - j\frac{\Omega}{a^2 + \Omega^2} \qquad \text{(ii)}$$

The real part is in the form of a Cauchy distribution, and we note that for $a > 0$

$$\int_{-\infty}^{\infty} \frac{a}{a^2 + \Omega^2} d\Omega = \pi$$

and that as $a \to 0$, $a/(a^2 + \Omega^2)$ becomes impulse-like, Therefore, as $a \to 0$

$$\frac{a}{a^2 + \Omega^2} \to \pi\delta(\Omega) \quad \text{and} \quad -j\frac{\Omega}{a^2 + \Omega^2} \to -j\frac{1}{\Omega}$$

so that we may define the Fourier transform of the unit-step function as

$$U_s(j\Omega) = \pi\delta(\Omega) + \frac{1}{j\Omega} \qquad \text{(iii)}$$

Note that the same result may be derived by considering that the unit-step as the running integral of the Dirac delta function, and using the integral property (Property 8 in Section 4.2) of the Fourier transform.

■ Example 15

Find the Fourier transform of the signum (sign) function

$$\text{sgn}(t) = \begin{cases} -1 & \text{for } t < 0 \\ 0 & \text{for } t = 0 \\ 1 & \text{for } t > 0 \end{cases} .$$

We note that

$$\text{sgn(x)} = 2u_s(t) - 1$$

and using the linearity property

$$\begin{aligned} \mathcal{F}\{\text{sgn}(t)\} &= 2\mathcal{F}\{u_s(t)\} - \mathcal{F}\{1\} \\ &= \left(2\pi\delta(\Omega) + 2\frac{1}{j\Omega}\right) - 2\pi\delta(\Omega) \\ &= \frac{2}{j\Omega} \end{aligned}$$

■ Example 16

Define the Fourier transform of the sinusoid $g(t) = \sin(\Omega_0 t)$.

In Example 11 the Fourier transform of a finite sinusoidal tone-burst of duration T

$$g_T(t) = \begin{cases} \sin \Omega_0 t & |t| < T/2 \\ 0 & \text{otherwise.} \end{cases}$$

was shown to be

$$G_T(j\Omega) = -j\frac{T}{2} \left\{ \frac{\sin\left((\Omega - \Omega_0)\,T/2\right)}{(\Omega - \Omega_0)\,T/2} - \frac{\sin\left((\Omega + \Omega_0)\,T/2\right)}{(\Omega + \Omega_0)\,T/2} \right\}. \tag{i}$$

Since $g(t) = \lim_{T \to \infty} g_T(t)$, we define $G(j\Omega) = \lim_{T \to \infty} G_T(j\Omega)$, and we note that

$$\int_{-\infty}^{\infty} \frac{\sin ax}{ax} dx = \frac{\pi}{a}.$$

As $a \to \infty$, $a \sin(ax)/ax$ becomes impulse-like with area π, so that

$$G(j\Omega) = \lim_{T \to \infty} G_T(j\Omega) = -j\pi \left(\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0) \right). \tag{ii}$$

The Fourier transform of a sinusoid is therefore purely imaginary, with a pair of impulses at plus and minus the frequency Ω_0.

A similar argument shows

$$\mathcal{F}\left\{\cos(\Omega_0 t)\right\} = \pi \left(\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0) \right)$$

which is a real function.

■ Example 17

Find the inverse Fourier transform of a one-sided spectrum $X(j\Omega) = u_s(\Omega)$, where $u_s(x)$ is the unit step function in x. Then $X(j\Omega) \equiv 0$ for $\Omega < 0$.

In Ex. 14 it is shown that

$$u_s(t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \pi\delta(\Omega) + \frac{1}{j\Omega}$$

Using the duality property, that is if $x(t) \overset{F}{\Longleftrightarrow} X(j\Omega)$ then $X(jt) \overset{F}{\Longleftrightarrow} 2\pi x(-\Omega)$, we have

$$X(jt) = \pi\delta(t) + \frac{1}{jt} \overset{F}{\Longleftrightarrow} 2\pi x(-\Omega) = 2\pi u_s(-\Omega).$$

Therefore

$$\frac{1}{2}\delta(-t) + \frac{j}{2\pi t} \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad u_s(\Omega) \qquad \text{(time reversal)}$$

$$\frac{1}{2}\delta(t) + \frac{j}{2\pi t} \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad u_s(\Omega) \qquad (\delta(t) \text{ assumed to be even).}$$

and

$$\mathcal{F}^{-1}\left\{u_s(\Omega)\right\} = \frac{1}{2}\delta(t) + \frac{j}{2\pi t}.$$

5   Fourier Transform Based Properties of Linear Systems

5.1   Response of Linear Systems to Aperiodic Inputs

The Fourier transform provides an alternative method for computing the response of a linear system to a transient input. Assume that a linear system with frequency response $H(j\Omega)$ is initially at rest, and is subsequently subjected to an aperiodic input $u(t)$ having a Fourier Transform $U(j\Omega)$. The task is to compute the response $y(t)$.

Assume that the input function $u(t)$, is a periodic function $u_p(t)$ with an arbitrarily large period T, and that since $u_p(t)$ is periodic it can be described by a set of complex Fourier coefficients $\{U_n\}$. The output $y_p(t)$ is periodic with period T and is described by its own set of Fourier coefficients $\{Y_n\}$. In Section 3.2 it was shown that the output coefficients are

$$Y_n = H(jn\Omega_0)U_n, \tag{53}$$

and that the Fourier synthesis equation for the output is

$$y_p(t) \quad = \quad \sum_{n=-\infty}^{\infty} Y_n e^{jn\Omega_0 t} \tag{54}$$

$$= \quad \sum_{n=-\infty}^{\infty} H(jn\Omega_0)U_n e^{jn\Omega_0 t} \tag{55}$$

$$= \quad \sum_{n=-\infty}^{\infty} H(jn\Omega_0) \left\{ \frac{\Omega_0}{2\pi} \int_{-T/2}^{T/2} u_p(t)e^{-jn\Omega_0 t}dt \right\} e^{jn\Omega_0 t} \tag{56}$$

As in the development of the Fourier transform, we let $T \to \infty$ so that $u_p(t) \to u(t)$, and in the limit write the summation as an integral

$$y(t) \quad = \quad \lim_{T\to\infty} y_p(t)$$

$$= \quad \int_{-\infty}^{\infty} H(j\Omega) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} u(t)e^{-j\Omega t}dt \right\} e^{j\Omega t}d\Omega$$

$$= \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\Omega)U(j\Omega)e^{j\Omega t}d\Omega. \tag{57}$$

This equation expresses the system output $y(t)$ in the form of the inverse Fourier transform of the product $H(j\Omega)U(j\Omega)$ or

$$y(t) = \mathcal{F}^{-1}\left\{H(j\Omega)U(j\Omega)\right\}. \tag{58}$$

We can therefore write

$$Y(j\Omega) = H(j\Omega)U(j\Omega), \tag{59}$$

which is the fundamental frequency-domain input/output relationship for a linear system. The output spectrum is therefore the product of the input spectrum and the system frequency response function:

Given a relaxed linear system with a frequency response $H(j\Omega)$ and an input that possesses a Fourier transform, the response may be found by the following three steps:

(1) Compute the Fourier transform of the input

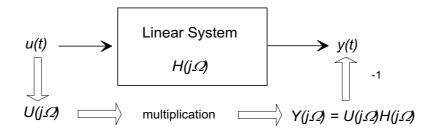$$U(j\Omega) = \mathcal{F}\left\{u(t)\right\}.$$

Figure 19: Frequency domain computation of system response.

(2) Form the output spectrum as the product

$$Y(j\Omega) = H(j\Omega)U(j\Omega),$$

(3) Compute the inverse Fourier transform

$$y(t) = \mathcal{F}^{-1}\left\{Y(j\Omega)\right\}.$$

Figure 19 illustrates the steps involved in computing the system response using the Fourier transform method.

■ **Example 18**

Use the Fourier transform method to find the response of a linear first-order system with a transfer function

$$H(s) = \frac{1}{\tau s + 1}$$

to a one-sided decaying exponential input

$$u(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t \geq 0. \end{cases}$$

where $a > 0$.

Solution: The frequency response of the system is

$$H(j\Omega) = \frac{1/\tau}{\tau + j\Omega} \tag{i}$$

and from Example 11 the Fourier transform of a decaying exponential input is

$$\begin{aligned} U(j\Omega) &= \mathcal{F}\left\{u(t)\right\} \\ &= \mathcal{F}\left\{e^{-a\tau}\right\} \\ &= \frac{1}{a + j\Omega}. \end{aligned} \tag{ii}$$

The output spectrum is the product of the transfer function and the frequency response

$$\begin{aligned} Y(j\Omega) &= H(j\Omega)U(j\Omega) \\ &= \frac{1/\tau}{1/\tau + j\Omega} \cdot \frac{1}{a + j\Omega}. \end{aligned} \tag{iii}$$

36

In order to compute the time domain response through the inverse transform, it is convenient to expand $Y(j\Omega)$ in terms of its partial fractions

$$Y(j\Omega) = \frac{1}{a\tau - 1}\left[\frac{1}{1/\tau + j\Omega} - \frac{1}{a + j\Omega}\right] \qquad \text{(iv)}$$

provided $a \neq 1/\tau$, and using the linearity property of the inverse transform

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left\{Y(j\Omega)\right\} \\ &= \frac{1}{a\tau - 1}\left[\mathcal{F}^{-1}\left\{\frac{1}{1/\tau + j\Omega}\right\} - \mathcal{F}^{-1}\left\{\frac{1}{a + j\Omega}\right\}\right] \end{aligned} \qquad \text{(v)}$$

Using the results of Example 11 once more

$$e^{-a\tau} \overset{\mathcal{F}}{\Longleftrightarrow} \frac{1}{a + j\Omega}, \qquad \text{(vi)}$$

the desired solution is

$$y(t) = \frac{1}{a\tau - 1}\left[e^{-t/\tau} - e^{-at}\right] \qquad \text{(vii)}$$

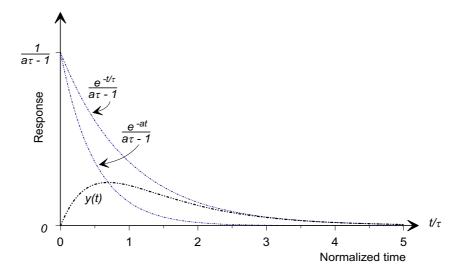These input/output relationships are summarized in Figure 20.

Figure 20: First-order system response to an exponential input

---

Before the 1960's frequency domain analysis methods were of theoretical interest, but the difficulty of numerically computing Fourier transforms limited their applicability to experimental studies. The problem lay in the fact that numerical computation of the transform of n samples of data required n^2 complex multiplications, which took an inordinate amount of time on existing digital computers. In the 1960's a set of computational algorithms, known as the Fast Fourier transform (FFT) methods, that required only $n \log_2 n$ multiplications for computing the Fourier transform of experimental data were developed. The computational savings are very great, for example in order to compute the transform of 1024 data points, the FFT algorithm is faster by a factor of more than 500. These computational procedures revolutionized spectral analysis and

frequency domain analysis of system behavior, and opened up many new analysis methods that had previously been impractical. FFT based system analysis is now routinely done in both software and in dedicated digital signal-processing (DSP) electronic hardware. These techniques are based on a "discrete-time" version of the continuous Fourier transforms described above, and have some minor differences in definition and interpretation.

5.2 The Frequency Response Defined Directly from the Fourier Transform

The system frequency response function $H(j\Omega)$ may be defined directly using the transform property of derivatives. Consider a linear system described by the single input/output differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y =$$
$$b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_1 \frac{du}{dt} + b_0 u \tag{60}$$

and assume that the Fourier transforms of both the input $u(t)$ and the output $y(t)$ exist. Then the Fourier transform of both sides of the differential equation may be found by using the derivative property (Property (7) of Section 4.2):

$$\mathcal{F}\left\{\frac{d^n f}{dt^n}\right\} = (j\Omega)^n F(j\Omega)$$

to give

$$\left\{a_n (j\Omega)^n + a_{n-1}(j\Omega)^{n-1} + \ldots + a_1(j\Omega) + a_0\right\} Y(j\Omega) =$$
$$\left\{b_m (j\Omega)^m + b_{m-1}(j\Omega)^{m-1} + \ldots + b_1(j\Omega) + b_0\right\} U(j\Omega), \tag{61}$$

which has reduced the original differential equation to an algebraic equation in $j\Omega$. This equation may be rewritten explicitly in terms of $Y(j\Omega)$ in terms of the frequency response $H(j\Omega)$

$$Y(j\Omega) = \frac{b_m(j\Omega)^m + b_{m-1}(j\Omega)^{m-1} + \ldots + b_1(j\Omega) + b_0}{a_n(j\Omega)^n + a_{n-1}(j\Omega)^{n-1} + \ldots + a_1(j\Omega) + a_0} U(j\Omega) \tag{62}$$
$$= H(j\Omega)U(j\Omega), \tag{63}$$

showing again the generalized multiplicative frequency domain relationship between input and output.

5.3 Relationship between the Frequency Response and the Impulse Response

In Example 9 it is shown that the Dirac delta function $\delta(t)$ has a unique property; its Fourier transform is unity for all frequencies
$$\mathcal{F}\left\{\delta(t)\right\} = 1,$$

The impulse response of a system $h(t)$ is defined to be the response to an input $u(t) = \delta(t)$, the output spectrum is then $Y_\delta(j\Omega) = \mathcal{F}\left\{h(t)\right\}$,

$$Y(j\Omega) = \mathcal{F}\left\{\delta(t)\right\} H(j\Omega)$$
$$= H(j\Omega). \tag{64}$$

or
$$h(t) = \mathcal{F}^{-1}\{H(j\Omega)\}. \tag{65}$$

In other words, the system impulse response $h(t)$ and its frequency response $H(j\Omega)$ are a Fourier transform pair:
$$h(t) \overset{\mathcal{F}}{\Longleftrightarrow} H(j\Omega). \tag{66}$$

In the same sense that $H(j\Omega)$ completely characterizes a linear system in the frequency response, the impulse response provides a complete system characterization in the time domain.

■ Example 19

An unknown electrical circuit is driven by a pulse generator, and its output is connected to a recorder for subsequent analysis, as shown in Figure 21.
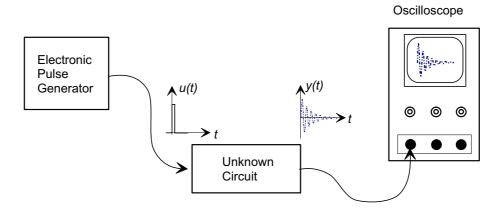

Figure 21: Setup for estimating the frequency response of an electrical circuit

The pulse generator produces pulses of 1 msec. duration and an amplitude of 10 volts. When the circuit is excited by a single pulse the output is found to be very closely approximated by a damped sinusoidal oscillation of the form

$$y(t) = 0.02e^{-t}sin(10t).$$

Estimate the frequency response of the system.

Solution: The input $u(t)$ is a short rectangular pulse, much shorter in duration than the observed duration of the system response. The impulse function $\delta(t)$ is the limiting case of a unit area rectangular pulse, as its duration approached zero. For this example we assume that the duration of the pulse is short enough to approximate a delta function, and because this pulse has an area of $10 \times 0.001 = 0.01$ v-s, we assume

$$u(t) = 0.01\delta(t) \tag{i}$$

and therefore assume that the observed response is a scaled version of the system impulse response,
$$y(t) = 0.01h(t), \tag{ii}$$

or

$$
\begin{aligned}
h(t) &= 100y(t) \\
&= 2e^{-5t} sin(12t).
\end{aligned}
$$

The frequency response is

$$
\begin{aligned}
H(j\Omega) &= \mathcal{F}\{h(t)\} \\
&= 2\mathcal{F}\left\{e^{-5t} sin(12t)\right\}. \qquad \text{(iii)}
\end{aligned}
$$

In Example 12 it is shown that

$$
\mathcal{F}\left\{e^{-\sigma t} \sin \Omega_0 t\right\} = \frac{\Omega_0}{(\sigma + j\Omega)^2 + \Omega_0^2},
$$

and substituting $\Omega_0 = 12$, $\sigma = 5$ gives

$$
H(j\Omega) = \frac{24}{(j\Omega)^2 + j20\Omega + 169} \qquad \text{(iv)}
$$

We therefore make the substitution $s = j\Omega$ and conclude that our unknown electrical network is a second-order system with a transfer function

$$
H(s) = \frac{24}{s^2 + 20s + 169}, \qquad \text{(v)}
$$

which has an undamped natural frequency $\Omega_n = 13$ radians/sec. and a damping ratio $\zeta = 10/13$. The input/output differential equation is

$$
\frac{d^2y}{dt^2} + 20\frac{dy}{dt} + 169y = 24u(t). \qquad \text{(vi)}
$$

■ Example 20

Find the impulse response of an all-pass filter that provides a phase shift of $-\pi/2$ rad. ($-90°$ deg.) at all frequencies. The required frequency response function is

$$
H(j\Omega) = -j\text{sgn}(\Omega) = \begin{cases} -j = e^{-j\pi/2} & \Omega > 0 \\ 0 & \Omega = 0 \\ j = e^{j\pi/2} & \Omega < 0 \end{cases}
$$

For example, if a signal $f(t) = \cos(\Omega_0(t)$ is applied to such a filter, the output $y(t)$ may be found

$$
f(t) = \cos(\Omega_o t) \stackrel{\mathcal{F}}{\Longleftrightarrow} \pi \left(\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right)
$$

and

$$
Y(j\Omega) = F(j\Omega)H(j\Omega) = -j\pi \left(\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)\right) = \mathcal{F}\{\sin(\Omega_0 t)\}
$$

and $y(t) = \sin(\Omega_0 t) = \cos(\Omega_0 t - \pi/2)$. Similarly if $f(t) = \cos(\Omega_0(t)$, then $y(t) = \sin(\Omega_0 t - \pi/2)$.

Solution: The impulse response is $h(t) = \mathcal{F}^{-1}\{H(j\Omega)\} = \mathcal{F}^{-1}\{-j\text{sgn}(\Omega)\}$. In Ex. 15 it was shown

$$\mathcal{F}\{\text{sgn}(t)\} = \frac{2}{j\Omega}$$

and using the duality property of the Fourier transform, that is if $f(t) \overset{\mathcal{F}}{\Longleftrightarrow} F(j\Omega)$, then $F(jt) \overset{\mathcal{F}}{\Longleftrightarrow} 2\pi f(-\Omega)$, we have

$$h(t) = \mathcal{F}\{-j\text{sgn}(\Omega)\} = \frac{1}{\pi t}$$

which is clearly acausal, and has a singularity at time $t = 0$. Although not physically realizable, this filter, known as a Hilbert transformer, plays an important role in communication theory. (Such filters may be approximated as digital filters using discrete time methods.)

5.4 The Convolution Property

A system with an impulse response $h(t)$, driven by an input $u(t)$, responds with an output $y(t)$ given by the convolution integral

$$\begin{aligned} y(t) &= h(t) \otimes u(t) \\ &= \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \end{aligned} \tag{67}$$

or alternatively by changing the variable of integration

$$y(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau. \tag{68}$$

In the frequency domain the input/output relationship for a linear system is multiplicative, that is $Y(j\Omega) = U(j\Omega)H(j\Omega)$. Because by definition

$$y(t) \overset{\mathcal{F}}{\Longleftrightarrow} Y(j\Omega),$$

we are lead to the conclusion that

$$h(t) \otimes u(t) \overset{\mathcal{F}}{\Longleftrightarrow} H(j\Omega)U(j\Omega). \tag{69}$$

The computationally intensive operation of computing the convolution integral has been replaced by the operation of multiplication. This result, known as the convolution property of the Fourier transform, can be shown to be true for the product of any two spectra, for example $F(j\Omega)$ and $G(j\Omega)$

$$\begin{aligned} F(j\Omega)G(j\Omega) &= \int_{-\infty}^{\infty} f(\nu)e^{-j\Omega\nu}d\nu . \int_{-\infty}^{\infty} g(\tau)e^{-j\Omega\tau}d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\nu)g(\tau)e^{-j\Omega(\nu+\tau)}d\tau d\nu, \end{aligned}$$

and with the substitution $t = \nu + \tau$

$$
\begin{aligned}
F(j\Omega)G(j\Omega) &= \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \right\} e^{-j\Omega t} dt \\
&= \int_{-\infty}^{\infty} (f(t) \otimes g(t)) \, e^{-j\Omega t} dt \\
&= \mathcal{F} \left\{ f(t) \otimes g(t) \right\}.
\end{aligned}
\tag{70}
$$

A *dual* property holds: if any two functions, $f(t)$ and $g(t)$, are multiplied together in the time domain, then the Fourier transform of their product is a convolution of their spectra. The dual convolution/multiplication properties are

$$
f(t) \otimes g(t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad F(j\Omega)G(j\Omega)
\tag{71}
$$

$$
f(t)g(t) \quad \overset{\mathcal{F}}{\Longleftrightarrow} \quad \frac{1}{2\pi} F(j\Omega) \otimes G(j\Omega).
\tag{72}
$$

■ Example 21

A real signal $f(t)$ has a two-sided spectrum $F(j\Omega)$. Find the signal $\hat{f}(t)$ that has a right-sided spectrum $\hat{F}(j\Omega)$, that is

$$
\hat{F}(j\Omega) = \begin{cases} F(j\Omega), & \Omega \geq 0 \\ 0, & \Omega < 0, \end{cases}
$$

or $\hat{F}(j\Omega) = u_s(\Omega)F(j\Omega)$, where $u(_s(\Omega)$ is the unit step function in Ω.

In Ex. 17 it is shown that

$$
\mathcal{F}^{-1} \left\{ u_s(\Omega) \right\} = \frac{1}{2}\delta(t) + \frac{j}{2\pi t}.
$$

If $\hat{F}(j\Omega)$ is formed as the product $\hat{F}(j\Omega) = u_s(\Omega)F(j\Omega)$, then

$$
\begin{aligned}
\hat{f}(t) &= \left(\frac{1}{2}\delta(t) + \frac{j}{2\pi t} \right) \otimes f(t) \\
&= \frac{1}{2}f(t) + \frac{j}{2\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau \\
&= \frac{1}{2} \left(f(t) + j\mathcal{H}\left\{ f(t) \right\} \right)
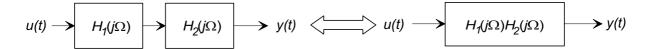\end{aligned}
$$

where

$$
\mathcal{H}\left\{ f(t) \right\} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau
$$

is known as the Hilbert transform, and and is used frequently in communication systems. The impulse response of the Hilbert transformer is

$$
h(t) = \frac{1}{\pi t}
$$

and because of the singularity at $t = 0$, the convolution should be written as a Cauchy principal value:

$$
\mathcal{H}\left\{ f(t) \right\} = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau
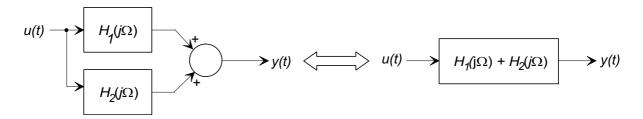$$

Figure 22: Frequency Response of Cascaded and Parallel Linear Systems

5.5 The Frequency Response of Interconnected Systems

If two linear systems $H_1(j\Omega)$ and $H_2(j\Omega)$ are connected in cascade, or series, as shown if Fig. 22 so that the output variable of the first is the input to the second, then provided the interconnection does not affect $y_1(t)$, overall frequency response is

$$
\begin{aligned}
Y_2(j\Omega) &= H_2(j\Omega)Y_1(j\Omega) \\
&= H_2(j\Omega)\{H_1(j\Omega)U(j\Omega)\} \\
&= \{H_2(j\Omega)H_1(j\Omega)\}U(j\Omega)
\end{aligned}
$$

(73)

The overall frequency response is therefore the product of the two cascaded frequency responses

$$H(j\Omega) = H_1(j\Omega)H_2(j\Omega). \tag{74}$$

Similarly, if two linear systems are connected in parallel so that their outputs are summed together, then

$$
\begin{aligned}
Y(j\Omega) &= H_1(j\Omega)U(j\Omega) + H_2(j\Omega)U(j\Omega) \\
&= \{H_1(j\Omega) + H_2(j\Omega)\}U(j\Omega),
\end{aligned}
$$

(75)

so that the overall frequency response is the sum of the two component frequency responses

$$H(j\Omega) = H_1(j\Omega) + H_2(j\Omega). \tag{76}$$

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