2.20 - Marine Hydrodynamics
Lecture 10

3.7 Governing Equations and Boundary Conditions for P-Flow

3.7.1 Governing Equations for P-Flow

(a) Continuity $\boxed{\nabla^2\phi = 0}$

(b) Bernoulli for P-Flow (steady or unsteady) $\boxed{p = -\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + gy\right) + C(t)}$

3.7.2 Boundary Conditions for P-Flow

Types of Boundary Conditions:

(c) Kinematic Boundary Conditions - specify the flow velocity $\vec{v}$ at boundaries. $\boxed{\frac{\partial\phi}{\partial n} = U_n}$

(d) Dynamic Boundary Conditions - specify force $\vec{F}$ or pressure p at flow boundary.

$$\boxed{p = -\rho\left(\phi_t + \frac{1}{2}\left(\nabla\phi\right)^2 + gy\right) + C\left(t\right) \text{ (prescribed)}}$$

The boundary conditions in more detail:

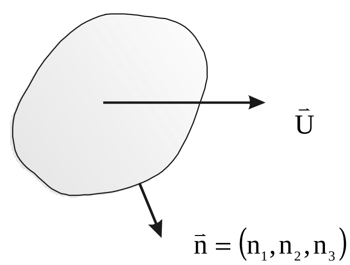
- Kinematic Boundary Condition on an impermeable boundary (no flux condition)

$$\underbrace{\vec{v}}_{\substack{\text{fluid velocity}\\ \vec{v}=\nabla\phi}} \cdot \hat{n} = \underbrace{\vec{U}}_{\text{boundary velocity}} \cdot \hat{n} = \underbrace{U_n}_{\text{nornal boundary velocity}} = \text{Given}$$

$$\nabla\phi \cdot \hat{n} = U_n \Rightarrow$$

$$(n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} + n_3 \frac{\partial}{\partial x_3})\phi = U_n \Rightarrow$$

$$\boxed{\frac{\partial \phi}{\partial n} = U_n}$$



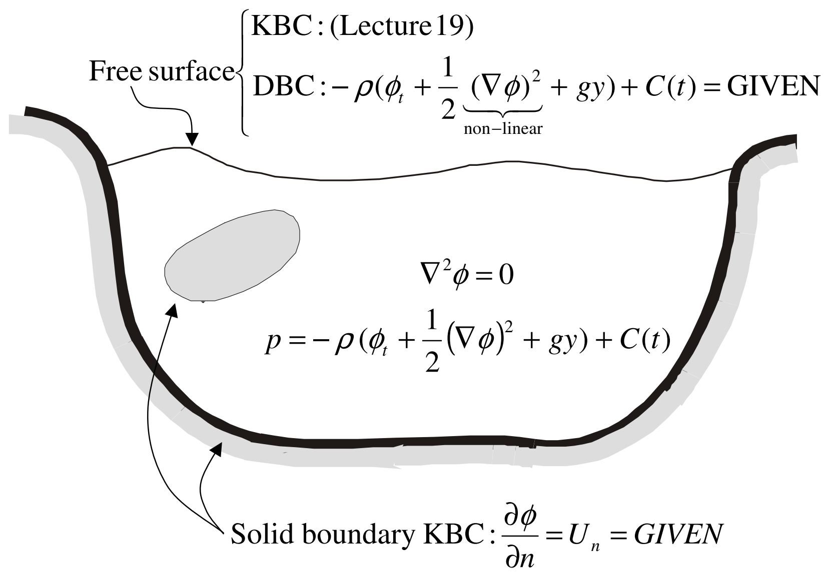
- Dynamic Boundary Condition: In general, pressure is prescribed

$$\boxed{p = -\rho \left(\phi_t + \frac{1}{2} \left(\nabla \phi \right)^2 + g y \right) + C\left(t\right) = \text{ Given}}$$

3.7.3 **Summary: Boundary Value Problem for P-Flow**

The aforementioned governing equations with the boundary conditions formulate the Boundary Value Problem (BVP) for P-Flow.

The general BVP for P-Flow is sketched in the following figure.

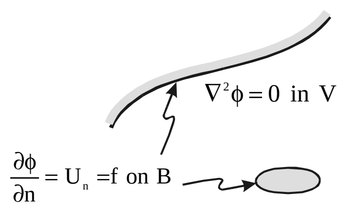


It must be pointed out that this BVP is satisfied **instantaneously**.

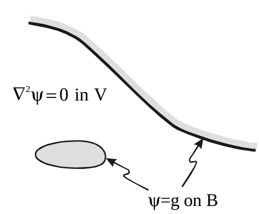
3.8 Linear Superposition for Potential Flow

In the **absence** of **dynamic boundary conditions**, the potential flow boundary value problem is **linear**.

- Potential function ϕ.



- Stream function ψ.



Linear Superposition: if $\phi_1, \phi_2, \ldots$ are harmonic functions, i.e., $\nabla^2\phi_i = 0$, then $\phi = \sum \alpha_i \phi_i$, where α_i are constants, are also harmonic, and is the solution for the boundary value problem provided the kinematic boundary conditions are satisfied, i.e.,

$$\frac{\partial\phi}{\partial n} = \frac{\partial}{\partial n}\left(\alpha_1\phi_1 + \alpha_2\phi_2 + \ldots\right) = U_n \text{ on } B.$$

The key is to combine known solution of the Laplace equation in such a way as to satisfy the kinematic boundary conditions (KBC).
The same is true for the stream function ψ. The K.B.C specify the value of ψ on the boundaries.

3.8.1 **Example**

Let $\phi_i\left(\vec{x}\right)$ denote a unit-source flow with source at $\vec{x}_i$, i.e.,

$$\begin{aligned}\phi_i\left(\vec{x}\right) \equiv \phi_{\text{source}}\left(\vec{x},\vec{x}_i\right) &= \frac{1}{2\pi}\ln\left|\vec{x}-\vec{x}_i\right| \qquad \text{(in 2D)}\\ &= -\left(4\pi\left|\vec{x}-\vec{x}_i\right|\right)^{-1} \text{ (in 3D)},\end{aligned}$$

then find m_i such that

$$\phi = \sum_i m_i\phi_i(\vec{x}) \;\; \text{satisfies KBC on B}$$

Caution: ϕ must be regular for $x \in V$, so it is required that $\vec{x} \notin V$.

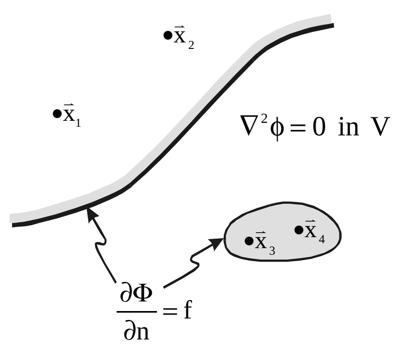


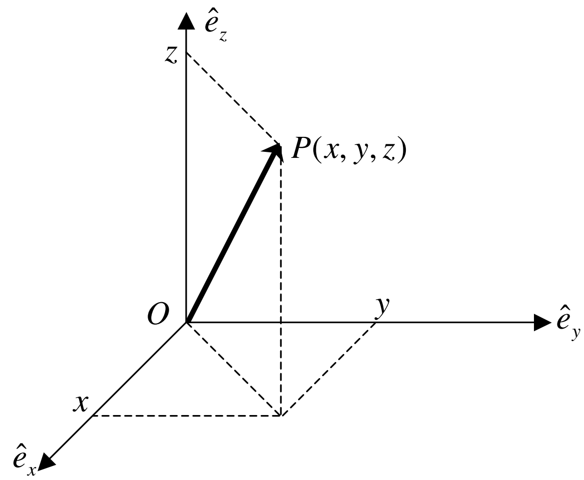
Figure 1: Note: $\vec{x}_j, j = 1,\ldots,4$ are *not* in the fluid domain V.

3.9 - Laplace equation in different coordinate systems (cf Hildebrand §6.18)

3.9.1 **Cartesian (x,y,z)**

$$\vec{v} = \left(\overset{\hat{i}}{u}, \overset{\hat{j}}{v}, \overset{\hat{k}}{w}\right) = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

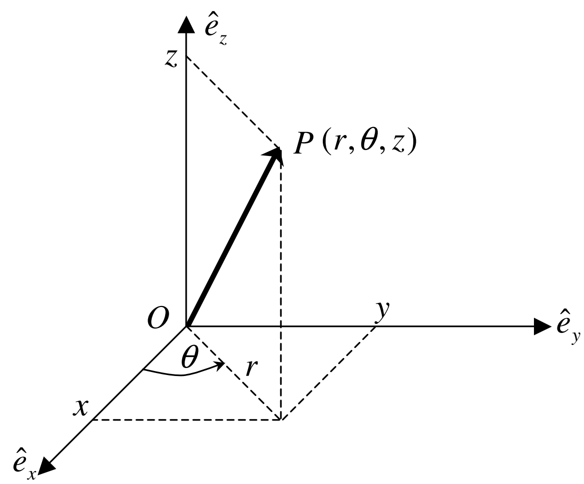


3.9.2 **Cylindrical (r,θ,z)**

$$r^2 = x^2 + y^2,$$
$$\theta = \tan^{-1}(y/x)$$

$$\vec{v} = \left(\overset{\hat{e}_r}{v_r}, \overset{\hat{e}_\theta}{v_\theta}, \overset{\hat{e}_z}{v_z}\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

$$\begin{aligned} \nabla^2\phi &= \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r}\frac{\partial\phi}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} \Leftrightarrow \\ \nabla^2\phi &= \frac{1}{r}\frac{\partial\phi}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} \end{aligned}$$

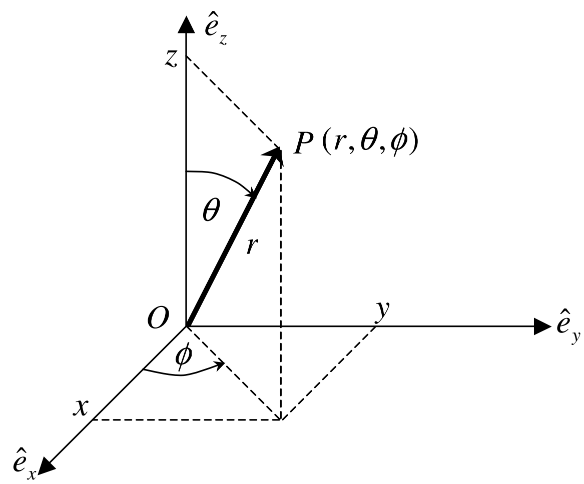


3.9.3 **Spherical (r,θ,φ)**

$$r^2 = x^2 + y^2 + z^2,$$
$$\theta = \cos^{-1}(z/r) \Leftrightarrow z = r\,(\cos\theta)$$
$$\varphi = \tan^{-1}(y/x)$$

$$\vec{v} = \nabla\phi = \left(\overset{\hat{e}_r}{v_r}, \overset{\hat{e}_\theta}{v_\theta}, \overset{\hat{e}_\varphi}{v_\varphi}\right) = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r(\sin\theta)}\frac{\partial\phi}{\partial\varphi}\right)$$

$$\nabla^2\phi = \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} \Leftrightarrow$$
$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$



3.10 Simple Potential flows

1. **Uniform Stream** $\nabla^2(ax + by + cz + d) = 0$

1D:	$\phi = Ux +$ constant	$\psi = Uy +$ constant;	$\vec{v} = (U, 0, 0)$
2D:	$\phi = Ux + Vy +$ constant	$\psi = Uy - Vx +$ constant;	$\vec{v} = (U, V, 0)$
3D:	$\phi = Ux + Vy + Wz +$ constant		$\vec{v} = (U, V, W)$

2. **Source (sink) flow**

2D, Polar coordinates

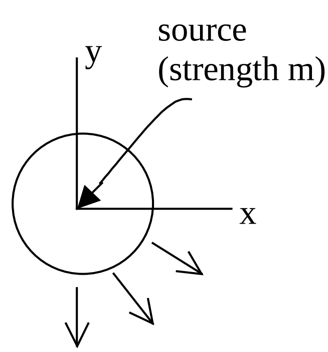
$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \text{ with } \ r = \sqrt{x^2 + y^2}$$

An axisymmetric solution: $\phi = a \ln r + b$. Verify that it satisfies $\nabla^2\phi = 0$, except at $r = \sqrt{x^2 + y^2} = 0$. Therefor, $r = 0$ must be excluded from the flow.

Define 2D source of strength m at r= 0:

$$\phi = \frac{m}{2\pi}\ln r$$

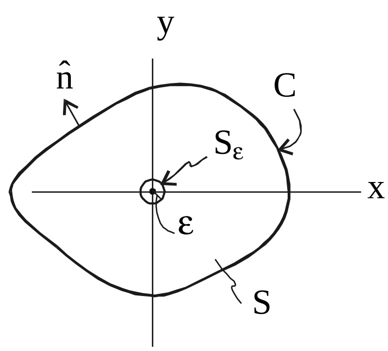
$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{e}_r = \frac{m}{2\pi r}\hat{e}_r \Longleftrightarrow v_r = \frac{m}{2\pi r}, \ v_\theta = 0$$



Net outward volume flux is

$$\oint_C \vec{v}\cdot\hat{n}ds = \iint_S \nabla\cdot\vec{v}ds = \iint_{S_\varepsilon} \nabla\cdot\vec{v}ds$$

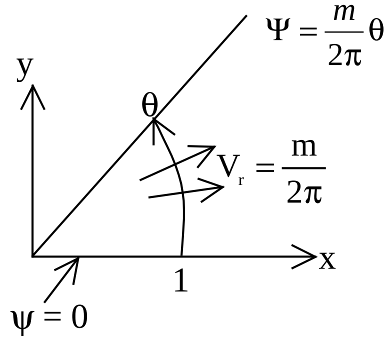
$$\oint_{C_\varepsilon} \vec{v}\cdot\hat{n}ds = \int_0^{2\pi} \underbrace{v_r}_{\frac{m}{2\pi r_\varepsilon}} r_\varepsilon d\theta = \underbrace{m}_{\substack{\text{source}\\\text{strength}}}$$



If $m < 0 \Rightarrow$ sink. Source m at (x_0, y_0):

$$\phi = \frac{m}{2\pi}\ln\sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$\phi = \frac{m}{2\pi}\ln r \text{ (Potential function)} \quad \longleftrightarrow \psi = \frac{m}{2\pi}\theta \quad \text{(Stream function)}$$



3D: Spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}, \cdots\right), \text{ where } r = \sqrt{x^2+y^2+z^2}$$

A spherically symmetric solution: $\phi = \dfrac{a}{r} + b$. Verify $\nabla^2\phi = 0$ except at $r = 0$.

Define a 3D source of strength m at $r = 0$. Then

$$\phi = -\frac{m}{4\pi r} \Longleftrightarrow v_r = \frac{\partial\phi}{\partial r} = \frac{m}{4\pi r^2},\ v_\theta = 0,\ v_\varphi = 0$$

Net outward volume flux is

$$\oint\!\!\!\!\!\!\iint v_r dS = 4\pi r_\varepsilon^2 \cdot \frac{m}{4\pi r_\varepsilon^2} = m \ (m < 0 \text{ for a sink })$$

3. **2D point vortex**

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Another particular solution: $\phi = a\theta + b$. Verify that $\nabla^2\phi = 0$ except at $r = 0$.

Define the potential for a point vortex of circulation Γ at $r = 0$. Then

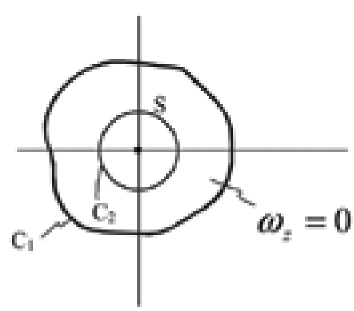
$$\phi = \frac{\Gamma}{2\pi}\theta \Longleftrightarrow v_r = \frac{\partial \phi}{\partial r} = 0,\ v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \text{ and,}$$
$$\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = 0 \text{ except at } \ r = 0$$

Stream function:

$$\psi = -\frac{\Gamma}{2\pi}\ln r$$

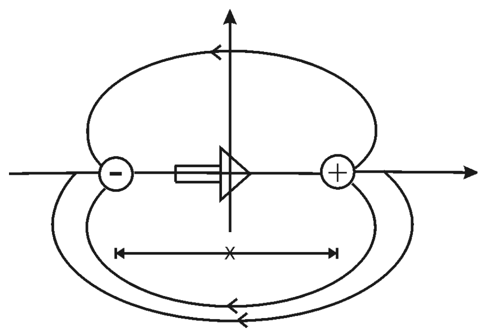
Circulation:

$$\int_{C_1} \vec{v}\cdot d\vec{x} = \int_{C_2} \vec{v}\cdot d\vec{x} + \underbrace{\int_{C_1 - C_2} \vec{v}\cdot d\vec{x}}_{\int\int_S \omega_z dS = 0} = \int_0^{2\pi} \frac{\Gamma}{2\pi r} r d\theta = \underbrace{\Gamma}_{\substack{\text{vortex}\\ \text{strength}}}$$



4. **Dipole (doublet flow)**

A **dipole** is a **superposition** of a **sink** and a **source** with the same strength.



2D dipole:

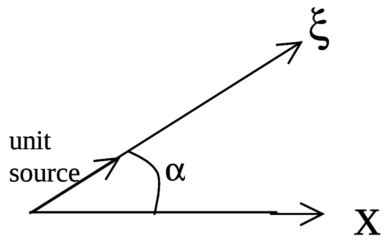
$$\phi = \frac{m}{2\pi}\left[\ln\sqrt{(x-a)^2+y^2} - \ln\sqrt{(x+a)^2+y^2}\right]$$

$$\lim_{a\to 0}\phi = \underbrace{\frac{\mu}{2\pi}}_{\substack{\mu = 2ma \\ \text{constant}}} \frac{\partial}{\partial\xi}\ln\sqrt{(x-\xi)^2+y^2}\bigg|_{\xi=0}$$

$$= -\frac{\mu}{2\pi}\frac{x}{x^2+y^2} = -\frac{\mu}{2\pi}\frac{x}{r^2}$$

2D dipole (doublet) of moment μ at the origin oriented in the +x direction.

NOTE: dipole $= \mu\frac{\partial}{\partial\xi}$ (unit source)



$$\phi = \frac{-\mu}{2\pi}\frac{x\cos\alpha + y\sin\alpha}{x^2+y^2} = \frac{-\mu}{2\pi}\frac{\cos\theta\cos\alpha + \sin\theta\sin\alpha}{r}$$

3D dipole:

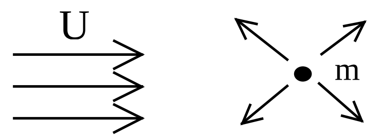
$$\phi = \lim_{a\to 0} \; -\frac{m}{4\pi}\left(\frac{1}{\sqrt{(x-a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+y^2+z^2}}\right) \text{ where } \; \mu = 2ma \text{ fixed}$$

$$= -\frac{\mu}{4\pi}\frac{\partial}{\partial\xi}\left.\frac{1}{\sqrt{(x-\xi)^2+y^2+z^2}}\right|_{\xi=0} = -\frac{\mu}{4\pi}\frac{x}{(x^2+y^2+z^2)^{3/2}} = -\frac{\mu}{4\pi}\frac{x}{r^3}$$

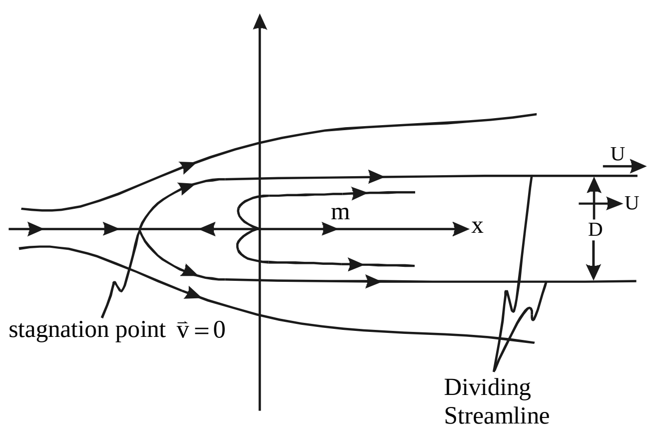
3D dipole (doublet) of moment μ at the origin oriented in the +x direction.

5. **<u>Stream and source: Rankine half-body</u>**

It is the **superposition** of a **uniform stream** of constant speed U and a **source** of strength m.



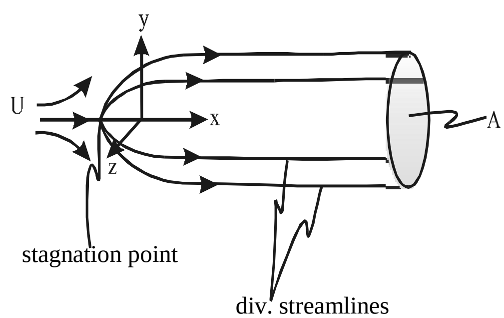
2D: $\phi = Ux + \frac{m}{2\pi} \ln \sqrt{x^2 + y^2}$



$$u = \frac{\partial \phi}{\partial x} = U + \frac{m}{2\pi} \frac{x}{x^2 + y^2}$$
$$u|_{y=0} = U + \frac{m}{2\pi x}, \;\; v|_{y=0} = 0 \Rightarrow$$
$$\vec{V} = (u, v) = 0 \text{ at } \;\; x = x_s = -\frac{m}{2\pi U}, \;\; y = 0$$

For large x, $u \to U$, and $UD = m$ by continuity $\Rightarrow D = \frac{m}{U}$.

3D: $\phi = Ux - \dfrac{m}{4\pi\sqrt{x^2+y^2+z^2}}$



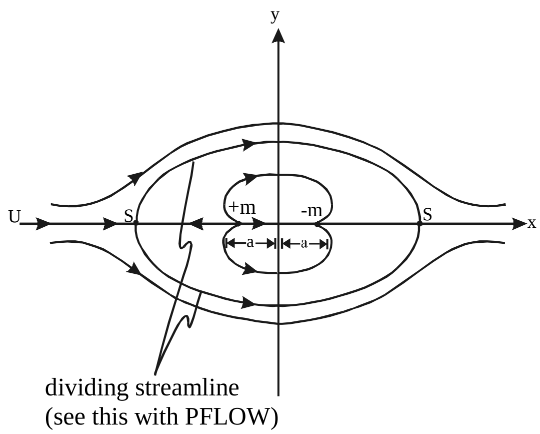
$$u = \frac{\partial \phi}{\partial x} = U + \frac{m}{4\pi}\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$u\big|_{y=z=0} = U + \frac{m}{4\pi}\frac{x}{|x|^3},\;\; v\big|_{y=z=0} = 0,\;\; w\big|_{y=z=0} = 0 \Rightarrow$$

$$\vec{V} = (u,v,w) = 0 \text{ at } x = x_s = -\sqrt{\frac{m}{4\pi U}},\;\; y = z = 0$$

For large x, $u \to U$ and $UA = m$ by continuity $\Rightarrow A = \dfrac{m}{U}$.

6. **<u>Stream + source/sink pair: Rankine closed bodies</u>**



To have a closed body, a necessary condition is to have $\sum m_{\text{in body}} = 0$

2D Rankine ovoid:

$$\phi = Ux + \frac{m}{2\pi}\left(\ln\sqrt{(x+a)^2 + y^2} - \ln\sqrt{(x-a)^2 + y^2}\right) = Ux + \frac{m}{4\pi}\ln\left(\frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}\right)$$

3D Rankine ovoid:

$$\phi = Ux - \frac{m}{4\pi}\left[\frac{1}{\sqrt{(x+a)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x-a)^2 + y^2 + z^2}}\right]$$

For Rankine Ovoid,

$$
\begin{aligned}
u =& \frac{\partial \phi}{\partial x} = U + \frac{m}{4\pi}\left[\frac{x+a}{\left((x+a)^2+y^2+z^2\right)^{3/2}} - \frac{x-a}{\left((x-a)^2+y^2+z^2\right)^{3/2}}\right] \\
u|_{y=z=0} =& U + \frac{m}{4\pi}\left[\frac{1}{(x+a)^2} - \frac{1}{(x-a)^2}\right] \\
=& U + \frac{m}{4\pi}\frac{(-4ax)}{(x^2-a^2)^2} \\
u|_{y=z=0} =& 0 \text{ at } \left(x^2-a^2\right)^2 = \left(\frac{m}{4\pi U}\right)4ax
\end{aligned}
$$

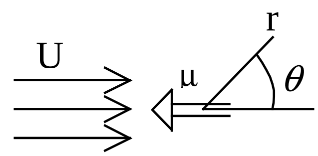
At $x = 0$,

$$u = U + \frac{m}{4\pi}\frac{2a}{(a^2+R^2)^{3/2}} \text{ where } \; R = y^2 + z^2$$

Determine radius of body R_0:

$$2\pi \int_0^{R_0} uRdR = m$$

7. **Stream + Dipole: circles and spheres**



2D: $\phi = Ux + \frac{\mu x}{2\pi r^2} \underset{x=r\cos\theta}{\underset{\uparrow}{=}} \cos\theta\left(Ur + \frac{\mu}{2\pi r}\right)$

The radial velocity is then

$$u_r = \frac{\partial\phi}{\partial r} = \cos\theta\left(U - \frac{\mu}{2\pi r^2}\right).$$

Setting the radial velocity $v_r = 0$ on $r = a$ we obtain $a = \sqrt{\frac{\mu}{2\pi U}}$. This is the K.B.C. for a stationary circle of radius a. Therefore, for

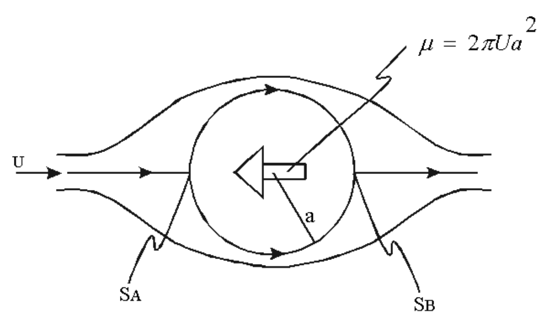
$$\mu = 2\pi U a^2$$

the potential

$$\phi = \cos\theta\left(Ur + \frac{\mu}{2\pi r}\right)$$

is **the** solution to ideal flow past a circle of radius a.

- *Flow past a circle* (U, a).



$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right)$$

$$V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta\left(1 + \frac{a^2}{r^2}\right)$$

$$V_\theta\big|_{r=a} = -2U\sin\theta \begin{cases} = 0 \text{ at } \ \theta = 0, \pi & \text{– stagnation points} \\ = \mp 2U \text{ at } \ \theta = \frac{\pi}{2}, \frac{3\pi}{2} & \text{– maximum tangential velocity} \end{cases}$$

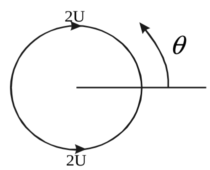
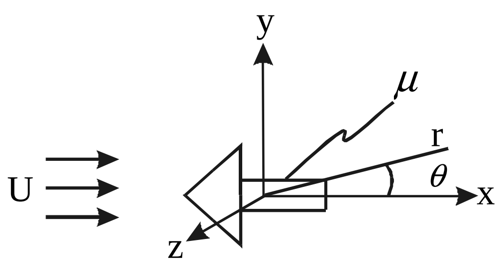


Illustration of the points where the flow reaches maximum speed around the circle.

3D: $\phi = Ux + \dfrac{\mu}{4\pi}\dfrac{\cos\theta}{r^2} = Ur\cos\theta\left(1 + \dfrac{\mu}{4\pi r^3}\right)$



The radial velocity is then

$$v_r = \frac{\partial\phi}{\partial r} = \cos\theta\left(U - \frac{\mu}{2\pi r^3}\right)$$

Setting the radial velocity $v_r = 0$ on $r = a$ we obtain $a = \sqrt[3]{\frac{\mu}{2\pi U}}$. This is the K.B.C. for a stationary sphere of radius a. Therefore, choosing

$$\mu = 2\pi U a^3$$

the potential

$$\phi = \cos\theta \left(Ur + \frac{\mu}{2\pi r}\right)$$

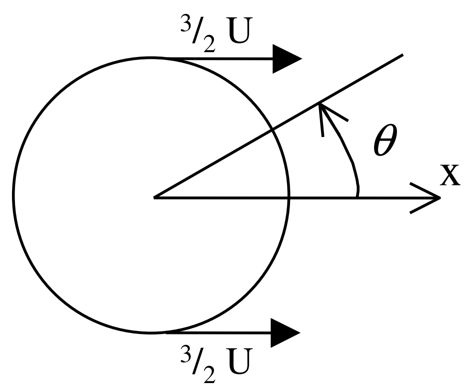
is **the** solution to ideal flow past a sphere of radius a.

• *Flow past a sphere* (U, a).

$$\phi = \mathrm{U}r\cos\theta\left(1 + \frac{a^3}{2r^3}\right)$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta\left(1 + \frac{a^3}{2r^3}\right)$$

$$v_\theta\,|_{r=a} = -\frac{3U}{2}\sin\theta \begin{cases} = 0 \text{ at } \theta = 0, \pi \\ = -\frac{3U}{2} \text{ at } \theta = \frac{\pi}{2} \end{cases}$$



8. **<u>2D corner flow</u>** Velocity potential $\phi = r^\alpha \cos\alpha\theta$; Stream function $\psi = r^\alpha \sin\alpha\theta$

(a) $\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\phi = 0$

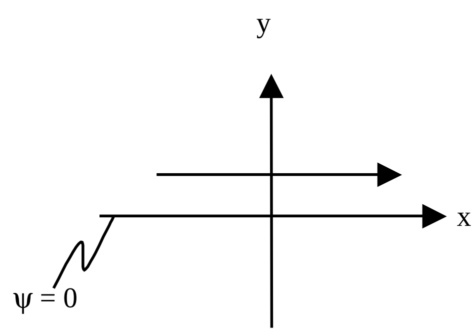
(b)

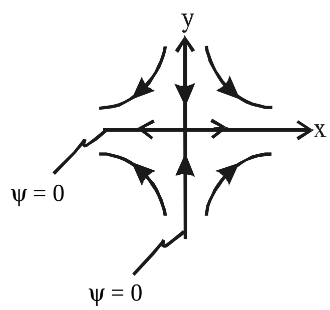
$$
\begin{aligned}
u_r &= \frac{\partial\phi}{\partial r} = \alpha r^{\alpha-1}\cos\alpha\theta \\
u_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\alpha r^{\alpha-1}\sin\alpha\theta \\
\therefore u_\theta &= 0\,\{\text{ or } \psi = 0\}\ \text{ on } \alpha\theta = n\pi, n = 0, \pm1, \pm2, \ldots
\end{aligned}
$$

i.e., on $\theta = \theta_0 = 0, \frac{\pi}{\alpha}, \frac{2\pi}{\alpha}, \ldots (\theta_0 \leq 2\pi)$

i. **Interior corner flow** – stagnation point origin: $\alpha > 1$. For example,

$\alpha = 1,\ \theta_0 = 0, \pi, 2\pi, \quad u = 1,\ v = 0$



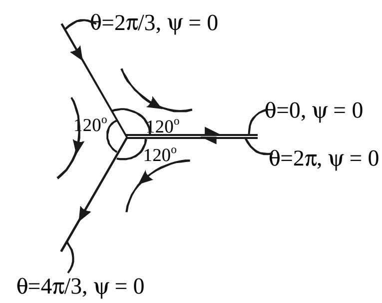


$\alpha = 2, \quad \theta_0 = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \quad u = 2x, v = -2y$

(90° corner)

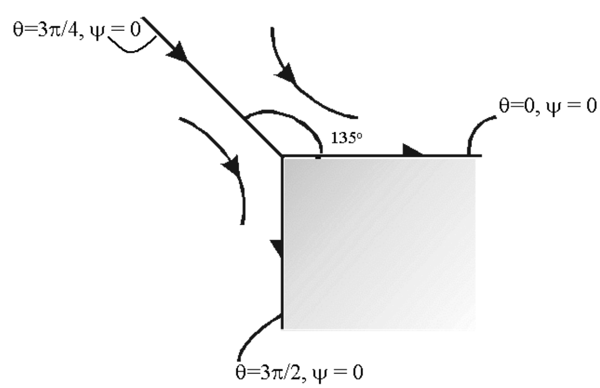
$\alpha = 3/2, \quad \theta_0 = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$

(120° corner)



$\alpha = 4/3, \quad \theta_0 = 0, \frac{3\pi}{4}, \frac{3\pi}{2}$ ~~$\left(\frac{9\pi}{4}\right)$~~

(135° corner)



ii. **Exterior corner flow**, $|v| \to \infty$ at origin:

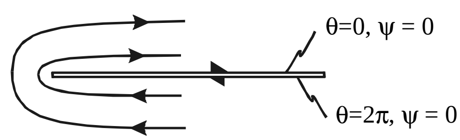
$$\begin{aligned} &\alpha < 1 \\ &\theta_0 = 0, \frac{\pi}{\alpha} \ \textbf{only} \end{aligned}$$

Since we need $\theta_0 \leq 2\pi$, we therefore require $\frac{\pi}{\alpha} \leq 2\pi$, i.e., $\alpha \geq 1/2$ only.

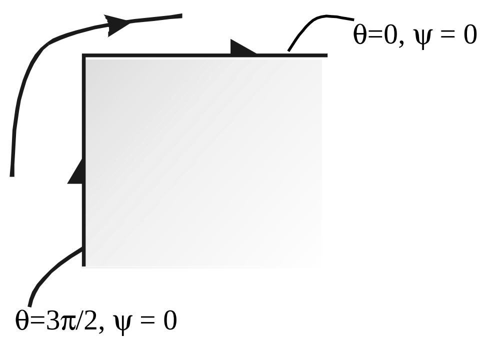
$$\begin{aligned} &1/2 \leq \alpha < 1 \\ &\theta_0 = 0, \frac{\pi}{\alpha} \end{aligned}$$

For example,

$\alpha = 1/2, \theta_0 = 0, 2\pi$ (½ infinite plate, flow around a tip)

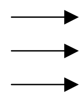
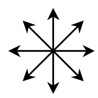
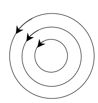
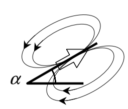


$\alpha = 2/3, \theta_0 = 0, \frac{3\pi}{2}$ (90^o exterior corner)



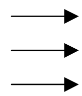
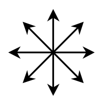
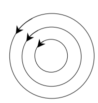
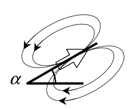
Appendix A1: Summary of Simple Potential Flows

Cartesian Coordinate System

Flow	Streamlines	Potential $\phi(x, y, z)$	Stream function $\psi(x, y)$
Uniform flow		$U_\infty x + V_\infty y + W_\infty z$	$U_\infty y - V_\infty x$
2D Source/Sink (m) at (x_o, y_o)		$\frac{m}{2\pi} \ln((x - x_o)^2 + (y - y_o)^2)$	$\frac{m}{2\pi} \arctan(\frac{y-y_o}{x-x_o})$
3D Source/Sink (m) at (x_o, y_o, z_o)	NA	$-\frac{m}{4\pi} \frac{1}{\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_o)^2}}$	NA
Vortex (Γ) at (x_o, y_o)		$\frac{\Gamma}{2\pi} \arctan(\frac{y-y_o}{x-x_o})$	$-\frac{\Gamma}{2\pi} \ln((x - x_o)^2 + (y - y_o)^2)$
2D Dipole (μ) at (x_o, y_o) at an angle α		$-\frac{\mu}{2\pi} \frac{(x-x_o)\cos\alpha+(y-y_o)\sin\alpha}{(x-x_o)^2+(y-y_o)^2}$	$\frac{\mu}{2\pi} \frac{(y-y_o)\cos\alpha+(x-x_o)\sin\alpha}{(x-x_o)^2+(y-y_o)^2}$
3D Dipole ($+x$) (μ) at (x_o, y_0, z_o)	NA	$-\frac{\mu}{4\pi} \frac{(x-x_o)}{((x-x_o)^2+(y-y_o)^2+(z-z_o)^2)^{3/2}}$	NA

Appendix A2: Summary of Simple Potential Flows

Cylindrical Coordinate System

Flow	Streamlines	Potential $\phi(r, \theta, z)$	Stream function $\psi(r, \theta)$
Uniform flow		$U_\infty r \cos\theta + V_\infty r \sin\theta + W_\infty z$	$U_\infty r \sin\theta - V_\infty r \cos\theta$
2D Source/Sink (m) at (x_o, y_o)		$\frac{m}{2\pi} \ln r$	$\frac{m}{2\pi}\theta$
3D Source/Sink (m) at (x_o, y_o, z_o)	NA	$-\frac{m}{4\pi r}$	NA
Vortex (Γ) at (x_o, y_o)		$\frac{\Gamma}{2\pi}\theta$	$-\frac{\Gamma}{2\pi} \ln r$
2D Dipole (μ) at (x_o, y_o) at an angle α		$-\frac{\mu}{2\pi} \frac{\cos\theta\cos\alpha + \sin\theta\sin\alpha}{r}$	$\frac{\mu}{2\pi} \frac{\sin\theta\cos\alpha + \cos\theta\sin\alpha}{r}$
3D Dipole $(+x)$ (μ) at (x_o, y_o, z_o)	NA	$-\frac{\mu}{4\pi} \frac{\cos\theta}{r^2}$	NA

Appendix A3: Combination of Simple Potential Flows

Stream + Source = Rankine *Half* Body	(2D) (3D)	$\phi = U_\infty x + \frac{m}{2\pi}\ln r$ $\quad x_s = -\frac{m}{2\pi U_\infty}$ $\quad D = \frac{m}{U_\infty}$ $\phi = U_\infty x - \frac{m}{4\pi}\frac{1}{\sqrt{x^2+y^2+z^2}}$ $\quad x_s = -\sqrt{\frac{m}{4\pi U_\infty}}$ $\quad A = \frac{m}{U_\infty}$
Stream + Source + Sink = Rankine *Closed* Body	(2D) (3D)	$\phi = U_\infty x + \frac{m}{2\pi}\left[\ln((x+a)^2+y^2) - \ln((x-a)^2+y^2)\right]$ $\phi = U_\infty x + \frac{m}{4\pi}(\frac{1}{\sqrt{(x+a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x-a)^2+y^2+z^2}})$
Stream + Dipole = Circle (Sphere) $R = a$	(2D) (3D)	$\phi = U_\infty x + \frac{\mu x}{2\pi r^2}$ $\quad$ if $\mu = 2\pi a^2 U_\infty$ $\quad \phi = U_\infty \cos\theta(r + \frac{a^2}{r})$ $\phi = U_\infty x + \frac{\mu\cos\theta}{4\pi r^2}$ $\quad$ if $\mu = 2\pi a^3 U_\infty$ $\quad \phi = U_\infty \cos\theta(r + \frac{a^3}{2r^2})$
2D Corner Flow	(2D)	$\phi = Cr^\alpha\cos(\alpha\theta)$ $\quad \psi = Cr^\alpha \sin(\alpha\theta)$ $\quad \theta_0 = 0, \frac{n\pi}{\alpha}$

Appendix B: Far Field Behavior of Simple Potential Flows

Far field behavior $r >> 1$		ϕ	$\vec{v} = \nabla\phi$
Source	(2D)	$\sim \ln r$	$\sim \frac{1}{r}$
	(3D)	$\sim \frac{1}{r}$	$\sim \frac{1}{r^2}$
Dipole	(2D)	$\sim \frac{1}{r}$	$\sim \frac{1}{r^2}$
	(3D)	$\sim \frac{1}{r^2}$	$\sim \frac{1}{r^3}$
Vortex	(2D)	~ 1	$\sim \frac{1}{r}$