13.42 Design Principles for Ocean Vehicles

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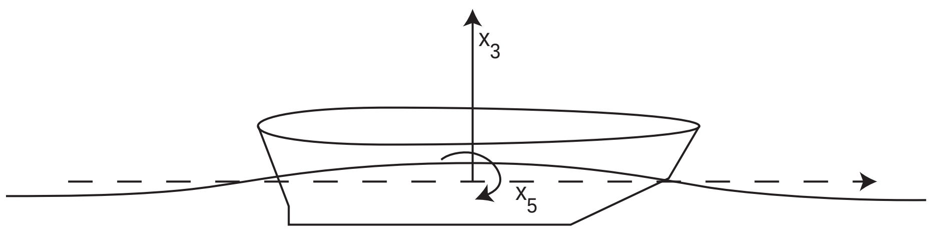
1. Coupled Equation of Motion in Heave and Pitch



Once we have set up the simple equation of motion for a vessel in heave it is natural to extend this discussion to the coupled heave-pitch equations of motion. Considering a ship floating on the free surface in waves. This ship will naturally heave and pitch due to the incident waves. It is not guaranteed that these two motions will be independent, however. Thus it becomes necessary to consider the motions together.

The body boundary condition must be properly specified, thus we need to know both the linear and angular velocities of the vessel:

$$\vec{V}_B = \frac{dx_3}{dt}\hat{k} + \vec{r} \times \left(\frac{dx_5}{dt}\hat{j}\right) = (z\,\dot{x}_5,\; \dot{x}_3 - x\,\dot{x}_5) \tag{1}$$

It follows that the velocity normal to the vessel hull is

$$\vec{V}_B \cdot \hat{n} = n_x\, z\, \dot{x}_5 + n_z(\dot{x}_3 - x\,\dot{x}_5) = n_z\,\dot{x}_3 + (n_x\, z - n_z\, x)\dot{x}_5 \tag{2}$$

where the position vector, $\vec{r}$, crossed with the unit outward normal, $\hat{n}$, is

$$\begin{pmatrix} n_4 \\ n_5 \\ n_6 \end{pmatrix} = \vec{r} \times \hat{n} \tag{3}$$

In general the potential functions do not change significantly from our previous cases. However it is necessary to account for the pitching motion in the radiation component of the total potential.

$$\phi(x,z,t) = \phi_I(x,z,t) + \phi_D(x,z,t) + \phi_R(x,z,t) \tag{4}$$

$$= Re\left\{a\left[\hat{\phi}_I(x,z) + \hat{\phi}_D(x,z)\right]e^{i\omega t} + \hat{x}_3\hat{\phi}_3(x,z)e^{i\omega t} + \hat{x}_5\hat{\phi}_5(x,z)e^{i\omega t}\right\}, \tag{5}$$

where the motions in the heave (3) and pitch (5) directions are

$$x_3(t) = Re\left\{\hat{x}_3 e^{i\omega t}\right\} \tag{6}$$

$$x_5(t) = Re\left\{\hat{x}_5 e^{i\omega t}\right\}. \tag{7}$$

The boundary conditions for the potentials, $\hat{\phi}_I$, $\hat{\phi}_D$, and $\hat{\phi}_3$ are the same as in the pure heave case. We must now, however, consider also the boundary conditions for the radiation potential due to pitching motions. Following the same formula from last lecture the equation of motion and boundary conditions at the free surface, sea floor and on the body are:

1. $\nabla^2 \hat{\phi}_5 = 0$
2. $-\omega^2 \hat{\phi}_5 + g\frac{\partial \hat{\phi}_5}{\partial z} = 0$
3. $\frac{\partial \hat{\phi}_5}{\partial z} = 0$ on $z = -H$
4. $\frac{\partial \hat{\phi}_5}{\partial n} = n_5$; on the body

In addition to the force in the vertical direction, we also need to formulate the pitching moment (about the y-axis) acting around the center of gravity. First we can re-write the vertical force dependent on the two motions as:

$$F_3(t) = \iint_{\bar{S}} \rho\, n_z \frac{\partial \phi_T}{\partial t} ds \tag{8}$$

$$= Re\left\{\iint_{\bar{S}} i\,\rho\,\omega\, e^{i\omega t} n_z \left(a\left[\hat{\phi}_I + \hat{\phi}_D\right] + \hat{x}_3\hat{\phi}_3 + \hat{x}_5\hat{\phi}_5\right) ds\right\} \tag{9}$$

$$= Re\left\{a\, e^{i\omega t}\left[\widehat{F}_{I3}+\widehat{F}_{D3}\right]+\hat{x}_3 e^{i\omega t}\left(\widehat{F}_{33}\right)+\hat{x}_5 e^{i\omega t}\left(\widehat{F}_{35}\right)\right\} \tag{10}$$

$$= Re\left\{a\, e^{i\omega t}\left[\widehat{F}_{I3}+\widehat{F}_{D3}\right]+\hat{x}_3 e^{i\omega t}\left(\omega^2 A_{33}-i\omega B_{33}\right)+\hat{x}_5 e^{i\omega t}\left(\omega^2 A_{35}-i\omega B_{35}\right)\right\} \tag{11}$$

Next, we can find F_5, the moment on the body due to pitching and heaving motion.

$$M_2 = F_5 = \int\int_{\bar{S}} \rho \frac{\partial \phi}{\partial t}(\vec{r}\times\hat{n})dS \tag{12}$$

where

$$F_5(t) = \int\int_{\bar{S}} \rho \frac{\partial \phi_T}{\partial t} n_5 \; dS \tag{13}$$

$$= Re\left\{a\, e^{i\omega t}\left[\widehat{F}_{I5}+\widehat{F}_{D5}\right]+\hat{x}_3 e^{i\omega t}\left(\widehat{F}_{53}\right)+\hat{x}_5 e^{i\omega t}\left(\widehat{F}_{55}\right)\right\} \tag{14}$$

$$= Re\left\{a\, e^{i\omega t}\left[\widehat{F}_{I5}+\widehat{F}_{D5}\right]+\hat{x}_3 e^{i\omega t}\left(\omega^2 A_{53}-i\omega B_{53}\right)+\hat{x}_5 e^{i\omega t}\left(\omega^2 A_{55}-i\omega B_{55}\right)\right\} \tag{15}$$

The hydrostatic forces in the two directions are

$$F_{3h} = -C_{33}\; x_3 - C_{35}\; x_5 \tag{16}$$

$$F_{5h} = -C_{53}\; x_3 - C_{33}\; x_5 \tag{17}$$

where the coefficients, C_{ij}, are based on the vessel geometry. Taking the pitch motion about the center of gravity (usually the c.g. is near midships), where $x_{cg} = 0$, we can write the equations of motions in heave and pitch:

$$m\,\frac{d^2 x_3}{dt^2} = F_3 + F_{3h} \tag{18}$$

$$I\,\frac{d^2 x_5}{dt^2} = F_5 + F_{5h} \tag{19}$$

The system of equations can now be written in matrix form assuming a harmonic input and an LTI system:

$$\begin{bmatrix} m+A_{33} & A_{35} \\ A_{53} & I+A_{55}\end{bmatrix}\begin{bmatrix}\ddot{x}_3 \\ \ddot{x}_5\end{bmatrix}+\begin{bmatrix} B_{33} & B_{35} \\ B_{53} & B_{55}\end{bmatrix}\begin{bmatrix}\dot{x}_3 \\ \dot{x}_5\end{bmatrix}+\begin{bmatrix} C_{33} & C_{35} \\ C_{53} & C_{55}\end{bmatrix}\begin{bmatrix}x_3 \\ x_5\end{bmatrix}= a\begin{bmatrix}\widehat{F}_{I3}+\widehat{F}_{D3} \\ \widehat{F}_{I5}+\widehat{F}_{D5}\end{bmatrix}e^{i\omega t} \tag{20}$$

Simply expressed in matrix notation, the system of equations for the coupled pitch and heave motions for a freely floating body can be rewritten as

$$\left[M + A\right]\underline{\ddot{x}} + \left[B\right]\underline{\dot{x}} + \left[C\right]\underline{x} = a\,\underline{\widehat{F}}\,e^{i\omega t} \tag{21}$$

where the vector motion is comprised of the linear heave motion, x_3, and the angular pitch motion, x_5.

$$\underline{x} = Re\left\{\begin{pmatrix} \hat{x}_3 \\ \hat{x}_5 \end{pmatrix} e^{i\omega t}\right\} \tag{22}$$

Writing the equation in matrix form allows us to better determine the amplitude response function (transfer function) between the forcing and the ship motions:

$$\underline{\hat{x}} = \frac{a}{\left\{-\omega^2\left[M + A\right] + i\omega\left[B\right] + \left[C\right]\right\}}\;\underline{\widehat{F}} \tag{23}$$

