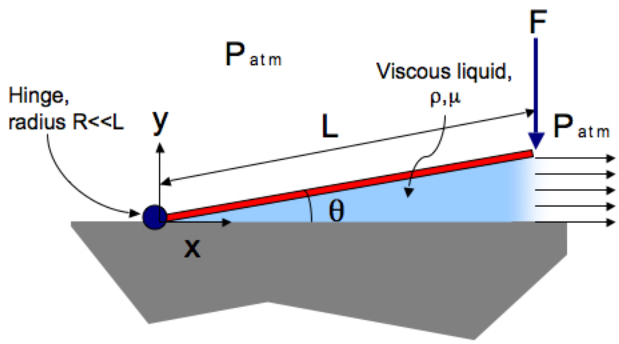
MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 6.20

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



A flat plate of breath L and length much greater than its breadth is attached to a plane floor by a hinge. The hinge has a radius R as shown. The plate is initially at a small angle θ_0 relative to the floor, and the region between it and the floor is filled with a viscous liquid. Starting at $t = 0$, the plate is forced toward the floor at a constant angular rate $-\frac{d\theta}{dt} = \omega$.

- Obtain an expression for the pressure distribution $p(x,t)$ under the plate in the limit of highly viscous (inertia-free) flow. The given quantities are L, R, θ, ω, ρ, μ, and the atmospheric pressure p_a outside the plate.
- Derive an expression for the vertically force $F_y^{tip}(t)$ which must be applied at the right-hand tip of the plate to make it close down at the specific constant angular rate.
- Write down the criteria which must be satisfied for your solutions to apply

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Solution:

- c) First, in order to realize an order of magnitude analysis of the terms in the N-S equations to show the validity of the lubrication approximation, let's gather more information about the order of magnitude of the velocities. First, let's use the mass conservation equation (by now this process should be quite familiar to you),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{6.20a}$$

and then dimensional analysis,

$$\frac{U_{avg}}{L} \quad \text{“}+\text{”} \quad \frac{V_{avg}}{\theta L} \quad \text{“}=\text{”} \quad 0, \tag{6.20b}$$

simplifying,

$$V_{avg} \quad \text{“}=\text{”} \quad \theta U_{avg}. \tag{6.20c}$$

Now, let's make an order of magnitude analysis. From the problem statement,

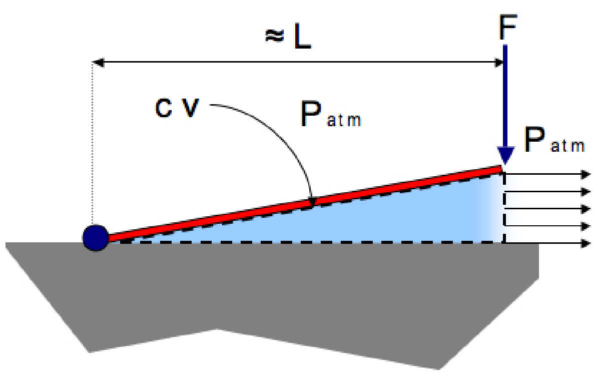
$$\theta \ll 1, \tag{6.20d}$$

$$R \ll L, \tag{6.20e}$$

then,

$$V_{avg} \ll U_{avg}. \tag{6.20f}$$

Now, using mass conservation again, assuming an incompressible liquid and using a triangular C.V. whose top surface moves with the hinged plate, and has a fixed base length (remember Shapiro 3.5),



$$\frac{\partial Vol}{\partial t} + \int_{CS} \underline{u}(x,t) \cdot \underline{n} dA = 0, \tag{6.20g}$$

then, the change in the C.V. volume must be equal to the flow coming out of the C.V..
Now using the C.V. geometry,

$$\frac{\partial Vol}{\partial t} = \frac{\omega x}{2} x = U_{avg}(x,t)\theta x = -\int_{CS} \underline{u}(x,t) \cdot \underline{n} dA, \tag{6.20h}$$

then,

$$U_{avg}(x,t) = \frac{\omega x}{2\theta}. \tag{6.20i}$$

Now, we have enough information to make an order of magnitude analysis. For the N-S equation in the x direction,

$$\underbrace{\rho\frac{\partial u}{\partial t}}_{\rho(\frac{\omega x}{2\theta})/(\frac{\theta}{\omega})} + \underbrace{\rho u\frac{\partial u}{\partial x}}_{\rho(\frac{\omega x}{2\theta})^2/x} + \underbrace{\rho v\frac{\partial u}{\partial y}}_{\rho(\frac{\omega x}{2\theta})^2/x} = -\frac{\partial p}{\partial x} + \mu\Bigg(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{(\frac{\omega x}{2\theta})/x^2} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{(\frac{\omega x}{2\theta})/(x^2\omega^2)}\Bigg). \tag{6.20j}$$

Then, the viscous stresses scale as

$$\frac{\mu\omega}{2\theta x}, \quad x \ direction \tag{6.20k}$$

$$\frac{\mu\omega}{2\theta^3 x}, \quad y \ direction \tag{6.20l}$$

then, the viscous stresses in the y direction are larger than those in the x direction. As a consequence, the right hand side terms (not considering the pressure) scale as $\frac{\mu\omega}{2\theta^3 x}$. On the other hand, the left hand side terms scale as $\frac{\rho\omega^2 x}{2\theta^2}$. Then, for viscous stresses to dominate,

$$\frac{\mu\omega}{2\theta^3 x} \gg \frac{\rho\omega^2 x}{2\theta^2}, \tag{6.20m}$$

then,

$$\frac{\omega x^2\theta}{\nu} \ll 1, \tag{6.20n}$$

for all x, or,

$$\frac{\omega L^2\theta}{\nu} \ll 1, \tag{6.20o}$$

since this could be the 'least viscous' point of the system. Notice that this non-dimensional number is a modified Re number, which can be easily visualized reordering,

$$\boxed{\tfrac{\rho(\omega L/\theta)(L\theta)}{\mu}\theta \ll 1}. \tag{6.20p}$$

- a) For Plane Poiseuille Flow (pressure driven channel flow),

$$\frac{\omega x^2}{2} = Q/depth = \left(-\frac{dP}{dx}\right)\frac{h^3}{12\mu}, \tag{6.20q}$$

then,

$$-\frac{dP}{dx} = \frac{6\mu\omega x^2}{h^3} = \frac{6\mu\omega}{\theta^3 x}, \tag{6.20r}$$

then, integrating to obtain an equation for pressure (assuming atmospheric pressure outside of the wedge),

$$\int_x^L\left(-\frac{dP}{dx}\right)dx = \int_x^L \frac{6\mu\omega}{\theta^3 x}dx, \tag{6.20s}$$

finally,

$$p(x,t) - p_{atm} = \frac{6\mu\omega}{\theta^3}\ln\left(\frac{L}{x}\right). \tag{6.20t}$$

In the last expression, $\theta = \theta_0 - \omega t$, then,

$$\boxed{p(x,t) = \tfrac{6\mu\omega}{(\theta_0-\omega t)^3}\ln\left(\tfrac{L}{x}\right) + p_{atm}.} \tag{6.20u}$$

In this equation, the pressure 'blows up' as the distance from the hinge is reduced, but is integrable and thus produces a finite net force upwards on the plate.

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- b) For $\theta \ll 1$, taking moments around the hinge, and assuming the hinge to be thin,

$$-F_y^{tip}(t)L = \int_0^L (p(x,t) - p_{atm})x dx \tag{6.20v}$$

$$= \frac{6\mu\omega}{\theta^3}\int_0^L \ln\left(\frac{L}{x}\right) x dx \tag{6.20w}$$

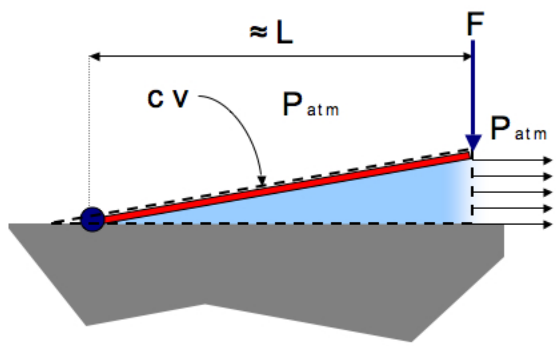
$$= \frac{6\mu\omega}{\theta^3}\int_0^1 z\ln\left(\frac{1}{z}\right) dz \tag{6.20x}$$

$$= \frac{3}{2}\frac{\mu\omega L^2}{\theta^3}. \tag{6.20y}$$

So,

$$\boxed{F_y^{tip}(t) = -\frac{3}{2}\frac{\mu\omega L}{\theta^3} = -\frac{3}{2}\frac{\mu\omega L}{(\theta_0-\omega t)^3},} \tag{6.20z}$$

where $F_y^{tip}(t)$ is the force per unit depth.



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Problem Solution by MC, Fall 2008

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