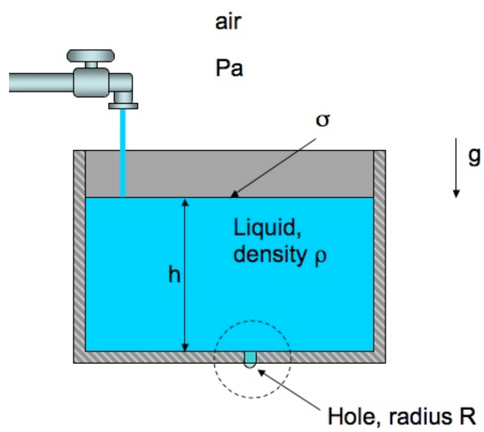
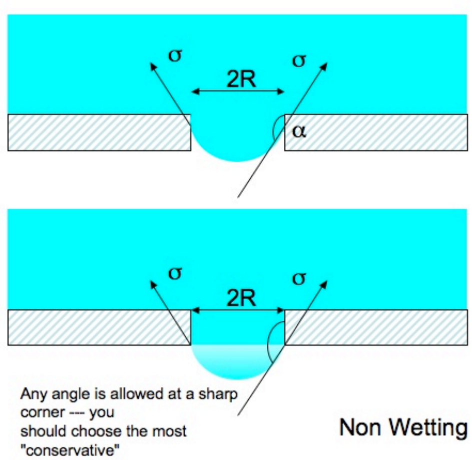
MIT Department of Mechanical Engineering

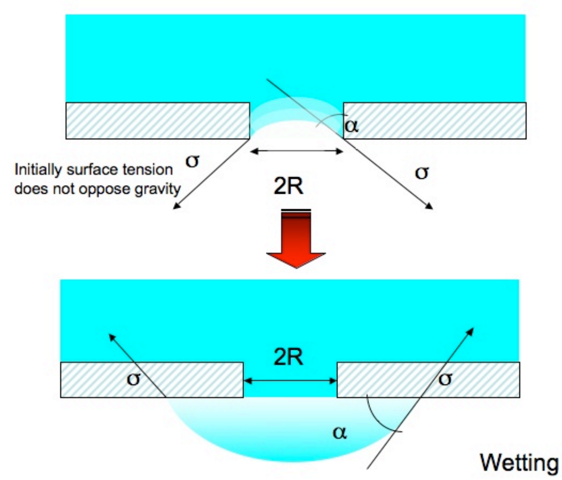
2.25 Advanced Fluid Mechanics

Problem 2.05

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin







A container is being filled with liquid of density ρ. A small, sharp-edged hole of radius R penetrates the container's bottom. The surface tension between the liquid and the ambient air is σ, and the contact angle for the air/liquid/container combination is α (measured from the wall through the liquid to the interface).

(a) Find the critical liquid depth h_c at which liquid first begins to flow through the hole in the bottom. Assume that $R \ll h$. (Hint Is the expression different depending on whether α is greater or smaller than $\pi/2$?)

(b) Evaluate h_c for the case when the liquid is water at $20°C$, $R = 0.1\,\mathrm{mm}$, $\sigma = 0.07\,\mathrm{N/m}$, and $\alpha = 120°$.

Solution:

First, let's assume that the Bond number is small, then surface tension effects dominate. Also, it is very important to understand the difference between the case $\alpha > \pi/2$ and $\alpha < \pi/2$.

For $\alpha > \frac{\pi}{2}$:

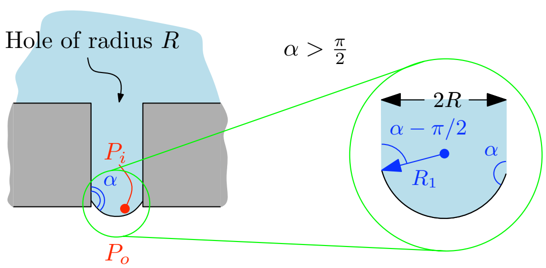
First let's analyze the case when the drop moves downwards through the hole starting at the container. Let's calculate the pressure inside the drop, no matter from where we approach the pressure has to be the same, as long as the drop is at rest (hydrostatics). From the bottom of the drop,

$$P_i - P_o = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

From the figure:

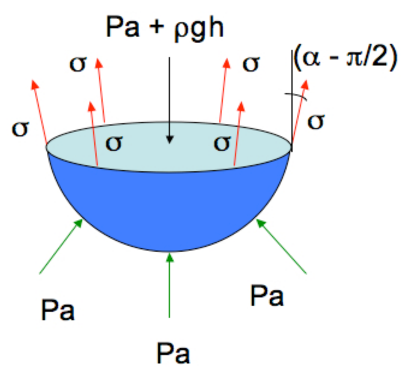
$$R_2 = R_1 = \frac{R}{\sin(\alpha - \pi/2)} = \frac{R}{-\cos(\alpha)}$$

From hydrostatics: $P_i = P_a + \rho g h$. Also, $P_o = P_a$.



$$\Rightarrow \qquad \rho g h + \cancel{P_a - P_a} = \sigma\left(\frac{2}{R/-\cos\alpha}\right)$$

$$\Rightarrow \boxed{h = \frac{\sigma}{\rho g R}(-\cos\alpha)}$$



Note that we could also have performed a force balance and obtain the same result. The force going down is simply the hydrostatic pressure force, $\pi R^2(\rho g h + P_a)$, and the forces going up are the atmospheric pressure force acting on the bottom of the drop, and the surface tension, $\pi R^2 P_a - 2\pi R\sigma\cos\alpha$. Now, performing a force balance,

$$\pi R^2 P_a - 2\pi R\sigma\cos\alpha = \pi R^2(\rho g h + P_a), \Rightarrow -2\sigma\cos\alpha = R(\rho g h), \Rightarrow h = -\frac{2\sigma\cos\alpha}{\rho g R}. \tag{2.05a}$$

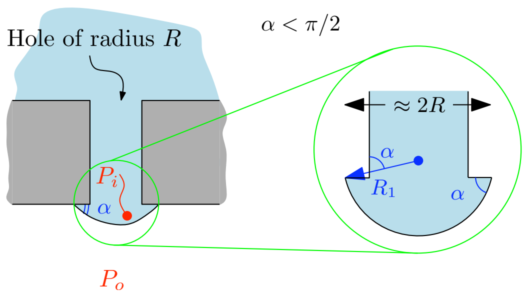
Now, when the drop reaches the bottom, any angle is allowed, then the factor $\cos\alpha$ is not a constrain, and the drop can resist higher hydrostatic pressures. The height reaches its maximum when $\cos\alpha = -1$, then,

$$\boxed{h_{max} = \tfrac{2\sigma}{\rho g R},} \tag{2.05b}$$

After this value, if the drop grows, the force to hold it up reduces and dropping starts. For water,

$$h_{max} = \frac{2(70[N/m]\times 10^{-3})}{10[\frac{m}{s}]1000\frac{kg}{m^3}1[m]\times 10^{-4}} = 0.14[m]. \tag{2.05c}$$

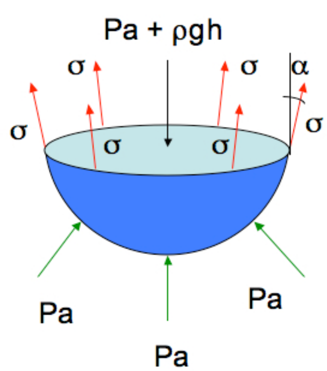
For $\alpha < \frac{\pi}{2}$:



Now, let's use the pressure argument again. From hydrostatics: $P_i = P_a + \rho g h$. Also, $P_o = P_a$, now calculating the pressure inside the drop from the bottom, $P_o - P_i = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ From the figure: $R_1 = R_2 = \frac{R'}{\sin\alpha}$

$$\Rightarrow \qquad \rho g h + \cancel{P_a - P_a} = \sigma\left(\frac{2}{R'/\sin\alpha}\right)$$

$$\Rightarrow \boxed{h = \frac{2\sigma}{\rho g R'}(\sin\alpha)}$$



Here too, we could also have performed a force balance and obtain the same result. The force going down is simply the hydrostatic pressure force, $\pi R'^2(\rho g h + P_a)$, and the forces going up are the atmospheric pressure force acting on the bottom of the drop, and the surface tension, $\pi R'^2 P_a + 2\pi R' \sigma \sin\alpha$. Now, performing a force balance,

$$\pi R'^2 P_a + 2\pi R' \sigma \sin\alpha = \pi R'^2(\rho g h + P_a), \Rightarrow 2\sigma\sin\alpha = R'(\rho g h), \Rightarrow h = \frac{2\sigma\sin\alpha}{\rho g R'}. \tag{2.05d}$$

□

Problem Solution by Tony Yu, MC (Updated), Fall 2008

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