2.25 – Fluid Mechanics
Overview of Lagrangian and Eulerian Descriptions

## Lagrangian Description

In the Lagrangian perspective, we follow fluid particles (material points) as they move through the flow. Mathematically, we specify the vector field

$$\underline{X}\left(\underline{X}_0,t\right) \tag{1}$$

where $\underline{X}$ is the position vector of a fluid particle at some time $t$ and $\underline{X}_0$ is the position vector of the particle at some reference time, say $t=t_0$. That is

$$\underline{X}_0=\underline{X}\left(\underline{X}_0,t=t_0\right) \tag{2}$$

We defined the Lagrangian velocity as the time derivative of the position vector following a fluid particle:

$$\underline{V}=\left.\frac{d\underline{X}}{dt}\right|_{\substack{\text{single}\\ \text{particle}}} \tag{3}$$

## Eulerian Description

On the other hand, the Eulerian perspective fixates on a particular point in space, and records the properties of the fluid elements passing through that point. Mathematically, specify

$$\underline{v}\left(\underline{x},t\right) \tag{4}$$

where $\underline{v}$ is the velocity vector at the laboratory coordinate $\underline{x}$ at time $t$.

One way to connect the Lagrangian and Eulerian descriptions is

$$\underbrace{\underline{v}\left(\underline{x}=\underline{X}\left(\underline{X}_0,t\right),t\right)}_{\text{Eulerian}}=\underbrace{\frac{d\underline{X}}{dt}\left(\underline{X}_0,t\right)}_{\text{Lagrangian}}. \tag{5}$$

Both sides of equation (5) describe the velocity of the fluid particle that was at $\underline{X}_0$ at $t_0$, but is at the position $\underline{x}$ at time $t$.

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## Material Derivative

The material derivative gives us a way to relate the Lagrangian description to the Eulerian one. It is defined for a quantity $F$ (which could be a scalar or a vector) as

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \left(\underline{v} \cdot \underline{\nabla}\right) F.$$

In Cartesian coordinates, this is written as

$$\frac{DF}{Dt} = \left.\frac{\partial F}{\partial t}\right|_{\substack{x,y,z \\ \text{fixed}}} + v_x \left.\frac{\partial F}{\partial x}\right|_{\substack{t,y,z \\ \text{fixed}}} + v_y \left.\frac{\partial F}{\partial y}\right|_{\substack{t,x,z \\ \text{fixed}}} + v_z \left.\frac{\partial F}{\partial z}\right|_{\substack{t,x,y \\ \text{fixed}}}$$

The material derivative can be used to obtain the ordinary differential equation describing the pathline of a particle. At any instant in time, the particle inhabits some laboratory point, described by $\underline{x}$. At some time $t$, the particle position $\underline{X}$ (in Lagrangian coordinates) is therefore equal to $\underline{x}$. Since the material derivative tells the Lagrangian time derivative in Eulerian coordinates, we can obtain the ODE we seek by setting

$$\left.\frac{d\underline{X}}{dt}\right|_{\substack{\text{single} \\ \text{particle}}} \equiv \frac{D\underline{x}}{Dt}. \tag{6}$$

The right-hand side can be thought of as the rate of change of the laboratory coordinate of the fluid particle of interest with time. Let us focus only on the $x$-coordinate of this particle. As shown in class, for this case the material derivative becomes

$$\left.\frac{dX}{dt}\right|_{\substack{\text{single} \\ \text{particle}}} = \frac{Dx}{Dt} = \left.\frac{\partial x}{\partial t}\right|_{\substack{x,y,z \\ \text{fixed}}} + v_x \left.\frac{\partial x}{\partial x}\right|_{\substack{t,y,z \\ \text{fixed}}} + v_y \left.\frac{\partial x}{\partial y}\right|_{\substack{t,x,z \\ \text{fixed}}} + v_z \left.\frac{\partial x}{\partial z}\right|_{\substack{t,x,y \\ \text{fixed}}} = 0 + v_x(1) + 0 + 0 = v_x. \tag{7}$$

Similarly, the time evolution of the $y$-coordinate of the particle is described by

$$\left.\frac{dY}{dt}\right|_{\substack{\text{single} \\ \text{particle}}} = \frac{Dy}{Dt} = \left.\frac{\partial y}{\partial t}\right|_{\substack{x,y,z \\ \text{fixed}}} + v_x \left.\frac{\partial y}{\partial x}\right|_{\substack{t,y,z \\ \text{fixed}}} + v_y \left.\frac{\partial y}{\partial y}\right|_{\substack{t,x,z \\ \text{fixed}}} + v_z \left.\frac{\partial y}{\partial z}\right|_{\substack{t,x,y \\ \text{fixed}}} = 0 + 0 + v_y(1) + 0 = v_y. \tag{8}$$