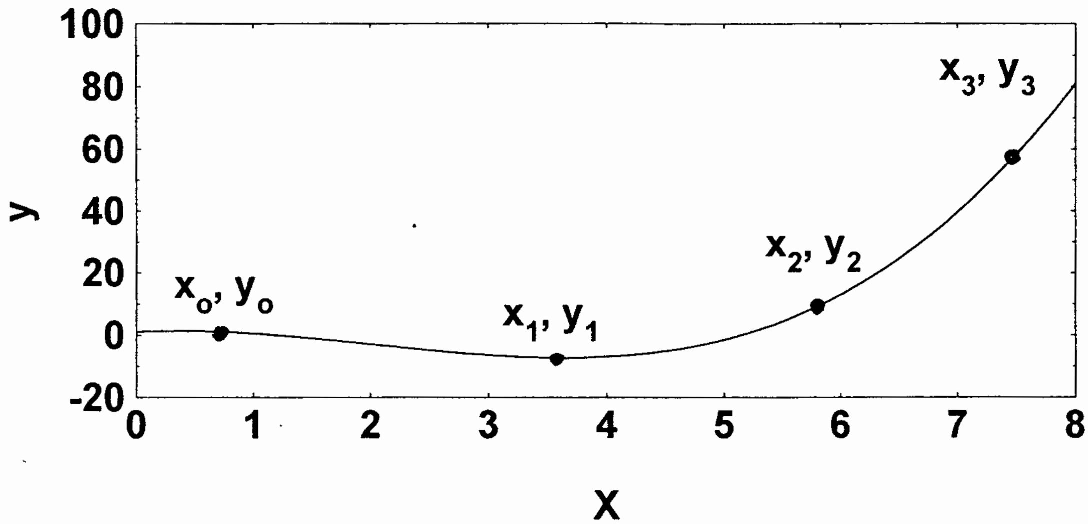
Curve Fitting and Interpolation

Polynomial Approximation to a Function



Suppose $y = f(x)$ where $f(x)$ is an unknown function. However, Suppose we have $N + 1$ pairs of values (x_k, y_k), $k = 0, 2, ..., N$. An approximation to $f(x)$ is the N^{th} order polynomial, $p_N(x)$ that passes through the $N+1$ points. This can be very useful. For example derivatives or integrals of f can be approximated by the corresponding derivatives or integrals of p_N. Also, p_N is an interpolating function for f.

One obvious way to determine the required $N + 1$ coefficients, c_i for p_N is to write the $N + 1$ equations:

$$\sum_{i=0}^{N} x_k^i c_i = y_k, \qquad k = 0, 1, ..., N$$

This is equivalent to the matrix equation, $\mathbf{X}_p \mathbf{c} = \mathbf{y}$

There is another way to determine an approximating (interpolating) polynomial that does not require solution of a matrix equation. To introduce it, suppose we seek the polynomial that passes through just two points (x_0, y_0), (x_1, y_1). It is easy to show that the polynomial is given by:

$$p_1(x) = \left(\frac{x - x_1}{x_0 - x_1}\right) y_0 + \left(\frac{x - x_0}{x_1 - x_0}\right) y_1$$

p is the linear combination of two order-1 polynomials L and can be written as:

$$p_1(x) = L_{1,0}(x) y_0 + L_{1,1}(x) y_1$$

The polynomials $L_{N,k}(x)$ are called *Lagrange Polynomials.* The polynomial representation can be extended to the case of $N+1$ points as:

$$p_N(x) = \sum_{k=0}^{N} L_{N,k} f(x_k)$$

The Lagrange Polynomials, $L_{N,k}(x)$ are polynomials of order N and have the following properties:

$$L_{N,k}(x_j) = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \end{cases}$$

The polynomials that have these properties are:

$$L_{N,k}(x) = \prod_{j=0,\, j\neq k}^{N} \frac{x - x_j}{x_k - x_j}$$

Lagrange Polynomials - Example

$y = f(x)$

x	y	k
0	1	0
1	2	1
3	1	2

$$P_2(x) = \sum_{k=0}^{2} L_{2k} y_k = L_{20} y_0 + L_{21} y_1 + L_{22} y_2$$

$$L_{20} = \frac{x - x_1}{x_0 - x_1} \frac{x - x_2}{x_0 - x_2} \qquad L_{21} = \frac{x - x_0}{x_1 - x_0} \frac{x - x_2}{x_1 - x_2}$$

$$L_{22} = \frac{x - x_0}{x_2 - x_0} \frac{x - x_1}{x_2 - x_1}$$

$$P_2(x) = 1 \frac{x - x_1}{x_0 - x_1} \frac{x - x_2}{x_0 - x_2} + 2 \frac{x - x_0}{x_1 - x_0} \frac{x - x_2}{x_1 - x_2} + 3 \frac{x - x_0}{x_2 - x_0} \frac{x - x_1}{x_2 - x_1}$$

$$P_2(x) = \frac{x-1}{-1} \frac{x-3}{-3} + 2 \frac{x}{1} \frac{x-3}{-2} + 1 \frac{x}{3} \frac{x-1}{2}$$

$$P_2(x) = \frac{1}{3}(x^2 - 4x + 3) - x^2 + 3x + \frac{1}{6}(x^2 - x)$$

at $x = 0$, $P_2(x) = 1$.

at $x = 1$, $P_2(x) = \frac{1}{3}(1 - 4 + 3) - 1 + 3 + \frac{1}{6}(1 - 1) = 2$

at $x = 3$, $P_2(x) = \frac{1}{3}(9 - 12 + 3) - 9 + 9 + \frac{1}{6}(9 - 3) = 1$

Numerical Differentiation

Numerical Differentiation is used when:

1. A functional form is so complicated that it is more convenient to do numerical integration,

2. when we have a table of values of $[x_i, f(x_i)]$ and we wish to find df/dx for some given value(s) of x.

Examples of situations for which derivatives are needed include:

1. Quantities given in terms of derivatives: $v = \frac{dx}{dt}$, $u = \frac{\partial \phi}{\partial x}$.
2. Mathematical procedures requiring derivatives:
   - A function $y = f(x)$ is to be approximated by $\hat{y} = \hat{f}(x)$ and $\hat{f}$ contains constants to be determined which minimize the error in the fit of the function to N points at x_i. Error $= \Sigma_1^N \left[\hat{f}(x_i) - f(x_i)\right]^2$
   - Finding the roots of $y = f(x)$. In other words, find the values of x such that $f(x) = 0$.

Two principal methods for obtaining numerical estimates of $f'(x_j)$ when we have a set (table) of pairs of values $[x_i,\ f(x_i) \equiv f_i]$, $i = 1, 2, ..., N$ are:

1. Develop relatively simple formulae that provide estimates of the derivative in terms of values of f_i and x_i,

2. Determine an analytic function $g(x)$ which is a good approximation to $f(x)$ and differentiate $g(x)$ analytically.

We will consider the first method here. The second is in the category of functional estimation or approximation.