1B.4 Contraction of a Newtonian Jet at Large Reynolds Numbers [OH].

a) Eq. of Continuity for the jet:

$$\int_V \left[\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \underset{\sim}{v}) \right] dV = 0 \qquad (1a)$$

Eq. of Motion:

$$\int_V \frac{\partial}{\partial t}(\rho \underset{\sim}{v})dV + \int_V (\nabla \cdot \rho \underset{\sim}{v}\underset{\sim}{v})dV + \int_V (\nabla \cdot \underset{\approx}{\pi})dV - \int_V \rho \underset{\sim}{g}\, dV = 0 \qquad (1b)$$

For steady-state, incompressible flow, the above equations become:

$$\int_V (\nabla \cdot \underset{\sim}{v})dV = 0 \qquad (2a)$$

$$\int_V (\nabla \cdot \rho \underset{\sim}{v}\underset{\sim}{v})dV + \int_V \nabla \cdot \underset{=}{\pi}\, dV - \int_V \rho \underset{\sim}{g}\, dV = 0 \qquad (2b)$$

Applying Divergence Theorem:

$$\int_S (\underset{\sim}{n} \cdot \underset{\sim}{v})dS = 0 \qquad (3a)$$

$$\int_S (\underset{\sim}{n} \cdot \rho \underset{\sim}{v}\underset{\sim}{v})dS + \int_S (\underset{\sim}{n} \cdot \pi)dS - \int_V \rho \underset{\sim}{g}\, dV = 0 \qquad (3b)$$

which are Eqs. (1B.4-1) & (1B.4-2)

At the free surface of the jet, $\underset{\sim}{n} \cdot \underset{\sim}{v} = 0$.
Thus, if $S_w \sim$ surface that is free; (3a) & (3b) become:

$$\int_{S_1} v_z \, dS - \int_{S_2} v_z \, dS = 0 \qquad (4a)$$

$$\int_{S_1} \rho v_z^2 \, dS - \int_{S_2} \rho v_z^2 \, dS + \int_{S_1} \pi_{zz} \, dS - \int_{S_2} \pi_{zz} \, dS - \int_{S_w} \pi_{zz} \, dS = 0 \quad (4b)$$

For (N), $\tau_{zz} = 0$ at surfaces $\rightarrow$ $\pi_{zz} = p$. The projection of S_w onto a plane of constant z is simply $(S_1 - S_2)$. If we define:

$$\langle \ldots \rangle_i := \int_{S_i} (\ldots) \, dS \Big/ \int_{S_i} dS \qquad ; \; i = 1, 2$$

Then, since $p|_{S_w} = p_a$; Eqs. (4a) & (4b) become:

$$\langle v_z \rangle_1 S_1 - v_2 S_2 = 0 \qquad (5a)$$

$$\rho \langle v_z^2 \rangle_1 S_1 - \rho v_2^2 S_2 + \langle \pi_{zz} \rangle_1 S_1 - p_a S_2 + p_a (S_2 - S_1) = 0 \quad (5b)$$

which are Eqs. (18.4-3) & (18.4-4)

b) If we assume $v_z|_{S_1} = k\left[1-\left(\frac{r}{R}\right)^2\right]$, then:

$$\frac{\langle v_z^2\rangle_1}{\langle v_z\rangle_1^2} = \frac{\int_0^R k^2\left[1-2\left(\frac{r}{R}\right)^2+\left(\frac{r}{R}\right)^4\right]2\pi r\,dr}{\left[\int_0^R k\left[1-\left(\frac{r}{R}\right)^2\right]2\pi r\,dr\right]^2/\pi R^2} = \frac{4}{3} \qquad (6)$$

From (5a), $v_2 = \frac{S_1\langle v_z\rangle_1}{S_2}$. Using this and the assumption that $\langle \pi_{zz}\rangle_1 \cong p_a$, we obtain from (5b):

$$S_1\langle v_z^2\rangle_1 - \langle v_z\rangle_1^2\,\frac{S_1^2}{S_2} = 0$$

Inserting (6) into this Eq. yields:

$$\frac{S_2}{S_1} = \frac{\langle v_z\rangle_1^2}{\langle v_z^2\rangle_1} = \frac{3}{4} \quad \text{, Q.E.D.}$$

1B.4 Contraction of a Newtonian Jet at Large Reynolds Numbers

a. The equations of continuity and motion for the jet can be written as follows:

$$\int_V \left[\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \underline{v}) \right] dV = 0 \qquad \text{(1a)}$$

$$\int_V \frac{\partial}{\partial t}(\rho \underline{v})\, dV + \int_V (\nabla \cdot \rho \underline{v}\underline{v})\, dV + \int_V (\nabla \cdot \underline{\underline{\Pi}})\, dV - \int_V (\rho g)\, dV = 0. \qquad \text{(1b)}$$

For steady-state, incompressible flow the above equations take the form,

$$\int_V (\nabla \cdot \underline{v})\, dV = 0 \qquad \text{(2a)}$$

$$\int_V (\nabla \cdot \rho \underline{v}\underline{v}) + \int_V (\nabla \cdot \underline{\underline{\Pi}})\, dV - \int_V (\rho g)\, dV = 0 \qquad \text{(2b)}$$

Applying Gauss's divergence theorem to eqn. (2a) and the first two terms of eqn. (2b) one obtains,

$$\int_S (\underline{n} \cdot \underline{v})\, dS = 0 \qquad \text{(3a)}$$

$$\int_S (\underline{n} \cdot \rho \underline{v}\underline{v})\, dS + \int_S (\underline{n} \cdot \underline{\underline{\Pi}})\, dS - \int_V (\rho g)\, dV = 0 \qquad \text{(3b)}$$

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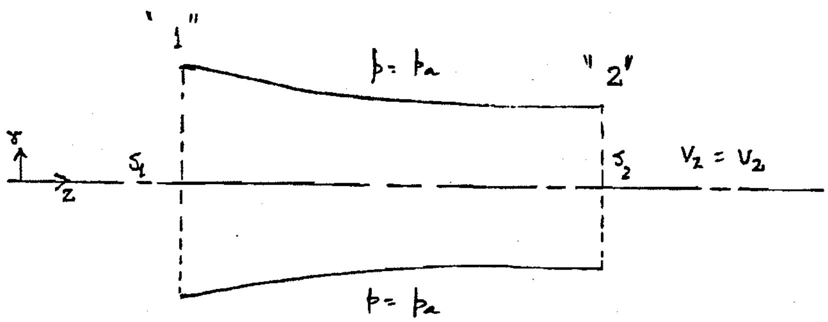

Along the free surface of the jet , $\underline{n} \cdot \underline{v} \equiv 0$

Applying eqs. (3a) and (3b) to the fluid contained between planes "1" and "2",

$$\int_{S_1} v_z \, dS - \int_{S_2} v_z \, dS = 0 \qquad \text{—(4a)}$$

$$\int_{S_1} \rho v_z^2 \, dS - \int_{S_2} \rho v_z^2 \, dS + \int_{S_1} \Pi_{zz} \, dS - \int_{S_2} \Pi_{zz} \, dS - \int_{\text{free surf}} \Pi_{zz} \, dS = 0 \qquad \text{—(4b)}$$

Define , $\langle v_z \rangle_a = \dfrac{\int_{S_a} v_z \, dS}{\int_{S_a} dS}$, $\langle v_z^2 \rangle_a = \dfrac{\int_{S_a} v_z^2 \, dS}{\int_{S_a} dS}$, $\langle \Pi_{zz} \rangle_a = \dfrac{\int_{S_a} \Pi_{zz} \, dS}{\int_{S_a} dS}$

$\Pi_{zz} = p + \tau_{zz}$: $\tau_{zz} = 0$ on all the surfaces because the fluid is Newtonian $\left(\frac{\partial v_z}{\partial z} = 0\right)$

The projected area of the free-surface in the 'δ_z direction' is $(S_1 - S_2)$.

Eqs (4) can now be written as

$$\langle v_z \rangle_1 S_1 - v_2 S_2 = 0 \qquad \text{—(5a)}$$

$$\rho \langle v_z^2 \rangle_1 S_1 - \rho v_2^2 S_2 + \langle \Pi_{zz} \rangle_1 S_1 - p_A S_2 + p_A (S_2 - S_1) = 0 \qquad \text{—(5b)}$$

The above are the same as eqs. (1B.4-3) and (1B4-4) in DPL.

Contd.

b (Contd.).

Assume that the flow is parabolic up to plane "1", i.e. at "1" v_z take the form,

$$(v_z)_1 = K\left[1 - \left(\frac{r}{R}\right)^2\right], \qquad -(6)$$

where K is a constant.

$$\frac{\langle v_z^2\rangle_1}{\langle v_z\rangle_1^2} = \frac{\int_0^R K^2\left[1 - 2\left(\frac{r}{R}\right)^2 + \left(\frac{r}{R}\right)^4\right] dr\,(2\pi r)}{\left[\int_0^R K\left[1 - \left(\frac{r}{R}\right)^2\right](2\pi r)\,dr\right]^2} = \frac{4}{3} \qquad -(7)$$

From eq. (5a) $\langle v_z\rangle_1 = \frac{v_2 S_2}{S_1}$. Substituting this into eq. (5b), assuming that $\langle \Pi_{zz}\rangle_1 = p_A$ we obtain,

$$S_1\langle v_z^2\rangle_1 - \langle v_z\rangle_1^2\,\frac{S_1^2}{S_2} = 0 \qquad -(8)$$

Substituting 7 into 8, we get

$$\boxed{\frac{S_2}{S_1} = \frac{3}{4}} \qquad -(9)$$

1B.5 Parallel-Disk Viscometer [OH]

a. Postulated flow field:

$$v_\theta = z f(r) \quad , \quad v_r = v_z = 0 \quad ; \quad \mathcal{P} = \mathcal{P}(r,z)$$

Continuity: $\nabla \cdot \underline{v} = \cancelto{0}{\frac{\partial v_r}{\partial r}} + \cancelto{0}{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0}{\frac{\partial v_z}{\partial z}} = 0$

Motion: r-comp.: $\frac{\partial \mathcal{P}}{\partial r} = \rho \frac{v_\theta^2}{r}$

θ-comp.: $0 = \mu \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2} \right]$

z-comp: $0 = \partial \mathcal{P} / \partial z$

b.

Substitute postulated form for v_θ into the θ-comp:

$$0 = \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2} + \frac{d^2 f}{dr^2} \rightarrow f(r) = A_1 r + \frac{1}{r} B_1$$

Apply B.C.s : 1) $v_\theta(z=B) = Wr$, 2) $v_\theta(z=0) = 0$

$\hookrightarrow B_1 = 0,\ A_1 = W/B$ $\quad \hookrightarrow$ ok

$$f(r) = \frac{W}{B} r \rightarrow v_\theta = \frac{W}{B} r z$$

c. $d\mathcal{T} = -\tau_{z\theta} r (2\pi r)\Big|_{z=B} dr = \mu \frac{\partial v_\theta}{\partial z} r (2\pi r)\Big|_{z=B} dr \quad \Big\| \int_{r=0}^{R} \ldots dr$

$$\mathcal{T} = \int_0^R \mu \frac{Wr}{B} 2\pi r^2 dr = \frac{\pi \mu W R^4}{2B}$$

1B.5 Parallel - Disk Viscometer

a. Postulate that for small W,

$$v_\theta = z\, f(r) \qquad \text{—(1a)}$$

$$v_r = v_z = 0 \qquad \text{—(1b)}$$

$$p = p(r, z) \qquad \text{—(1c)}$$

The equation of continuity becomes:

$$\nabla \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Therefore, a solution of the form (1) satisfies continuity exactly. The components of the equation of motion can be written as follows:

r- component: $$(\partial P/\partial r) = \rho(v_\theta^2/r) \qquad \text{—(2a)}$$

θ- component: $$0 = \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{\partial^2 v_\theta}{\partial z^2}\right] \qquad \text{—(2b)}$$

z- component: $$(\partial P/\partial z) = 0 \qquad \text{—(2c)}$$

b. Substituting for v_θ from (1a) in eq. (2b), we get the equidimensional ode,

$$f'' + \frac{f'}{r} - \frac{f}{r^2} = 0 \qquad \text{—(3a)}$$

$$\therefore f(r) = A_1 r + \frac{B_1}{r} \qquad \text{—(3b)}$$

The boundary conditions for v_θ are:

$$v_\theta\,(z = B) = W r \qquad \text{—(4a)}$$

$$v_\theta\,(z = 0) = 0 \qquad \text{—(4b)}$$

Written in terms of $f(r)$, (4a) becomes

$$f(r) = \frac{W}{B} r \qquad \text{—(5)}$$

Add[illegible] v_θ (and f) must be bounded at $r = 0$.

$\therefore A_1 = \frac{W}{B}$; $B_1 = 0$ $\Rightarrow$ $\boxed{v_\theta = \frac{zWr}{B}}$

torque required to turn the upper disk.

$$\theta \; r \, dr \, (2\pi r)$$

$$d\mathcal{T} = \mu \frac{\partial V_\theta}{\partial z} \, r \, dr \, (2\pi r)$$

$$\mathcal{T} = \int_0^R \left[\mu \frac{W r}{B} \; 2\pi r^2 \right] dr = \underline{\underline{\frac{\pi \mu W R^4}{2B}}}$$

1B.7 Steady Simple Elongational Flow and Elongational Viscosity [JDS]

a) Eqn. of Cont. : $\nabla \cdot \underset{\sim}{v} = 0$

$$\nabla \cdot \underset{\sim}{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

$$= -\frac{1}{2}\dot{\varepsilon}\frac{1}{r}\frac{\partial}{\partial r}(r^2) + 0 + \dot{\varepsilon} = 0 \text{ , OK}$$

$$\dot{\gamma} = \nabla \underset{\sim}{v} + (\nabla \underset{\sim}{v})^{\dagger} = \begin{bmatrix} -\dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon} & 0 \\ 0 & 0 & 2\dot{\varepsilon} \end{bmatrix} \text{ in cylindrical coordinates}$$

N : $\tau = -\mu \dot{\gamma} \longrightarrow \tau_{rr} = \tau_{\theta\theta} = \mu \dot{\varepsilon}$; $\tau_{zz} = -2\mu\dot{\varepsilon}$

b) Eqn. of Motion:

$$\cancel{\frac{\partial}{\partial t}(\rho \underset{\sim}{v})} = -\cancel{[\nabla \cdot \rho \underset{\sim}{v}\underset{\sim}{v}]} - [\nabla \cdot \underset{\approx}{\pi}] + \cancel{\rho \underset{\sim}{g}} \rightarrow \text{no ext. forces}$$

steady, incomp. neglect

$$\nabla \cdot \underset{\approx}{\pi} = \underset{\sim}{0} \rightarrow \underset{\approx}{\pi} \sim \text{constant}$$

At free surface : $\underset{\approx}{\pi} \cdot \underset{\sim}{\delta}_r = p_0 \underset{\sim}{\delta}_r$ [B.C.]

$$\therefore \pi_{rr} = p_0 \rightarrow \tau_{rr} + p = p_0$$

c) $\pi_{zz} = p + \tau_{zz} \rightarrow \pi_{zz} - p_0 = p + \tau_{zz} - \tau_{rr} - p$

$$= \tau_{zz} - \tau_{rr}$$

$$= -3\mu(\partial v_z/\partial z) \;//$$

1B.7 a) <u>Eq. of Cont.</u> $\nabla \cdot \underset{\sim}{v} = 0$

$$\nabla \cdot \underset{\sim}{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

$$= -\frac{1}{2}\dot{\varepsilon}\,\frac{1}{r}\frac{\partial}{\partial r}(r^2) + 0 + \dot{\varepsilon} = 0 \quad /\text{QED}$$

$$\underset{=}{\dot{\gamma}} = \nabla v + (\nabla v)^{+} = \begin{bmatrix} -\dot{\varepsilon} & 0 & 0 \\ 0 & -\dot{\varepsilon} & 0 \\ 0 & 0 & 2\dot{\varepsilon} \end{bmatrix}$$ <u>from Table A.7-2</u>

[Cyl. Coord.]

$$\underset{=}{\tau} = -\mu \underset{=}{\dot{\gamma}} \;\; \text{(N)} \longrightarrow \boxed{\tau_{rr} = \tau_{\theta\theta} = \mu\dot{\varepsilon}, \;\; \tau_{zz} = -2\mu\dot{\varepsilon}}$$

b) <u>Eq. of Motion:</u>

$$\cancel{\frac{\partial}{\partial t}(\rho \underset{\sim}{v})} = -[\nabla \cdot \rho \cancel{\underset{\sim}{v}\underset{\sim}{v}}] - \boxed{\nabla \cdot \underset{=}{\pi}} + \cancel{\rho \underset{\sim}{g}}$$

Steady, incomp. — neglect — no ext. forces

$$\nabla \cdot \underset{=}{\pi} = \underset{=}{0} \longrightarrow \underset{=}{\pi} \sim \text{constant}$$

At the free surface : $\underset{=}{\pi} \cdot \underset{\sim}{\delta}_r = p_0 \underset{\sim}{\delta}_r$ (Boundary Cond.)

$$\pi_{rr} = p_0$$

$$\tau_{rr} + p = p_0 \quad //$$

c) $\pi_{zz} = p + \tau_{zz}$

$$\pi_{zz} - p_0 = p + \tau_{zz} - \tau_{rr} - p = \tau_{zz} - \tau_{rr}$$

$$= -3\mu\dot{\varepsilon}$$

$$= -3\mu\frac{\partial v_z}{\partial z}, \quad \text{QED}$$