MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.710 Optics **Spring '09**

Solutions to Problem Set #6 Due Wednesday, Apr. 15, 2009

Problem 1: Grating with tilted plane wave illumination

1. a) In this problem, one–dimensional geometry along the x–axis is considered. The Fresnel diffraction pattern, the field just behind the grating illuminated by the plane wave, is

$$g_+(x, z = 0) = g_t(x) g_-(x, z = 0) = \exp\left\{i\frac{m}{2}\sin\left(2\pi\frac{x}{\Lambda}\right)\right\}\exp\left\{i\frac{2\pi}{\lambda}\theta x\right\}. \quad (1)$$

Note that the transmission function can be expanded as

$$g_t(x) = \exp\left\{i\frac{m}{2}\sin\left(2\pi\frac{x}{\Lambda}\right)\right\} = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\exp\left\{iq\frac{2\pi}{\Lambda}x\right\}. \quad (2)$$

Using eq. (2), we can rewrite eq. (1) as

$$g_+(x, z = 0) = \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\exp\left\{i\frac{2\pi}{\lambda}\left(\theta + \frac{q\lambda}{\Lambda}\right)x\right\}. \quad (3)$$

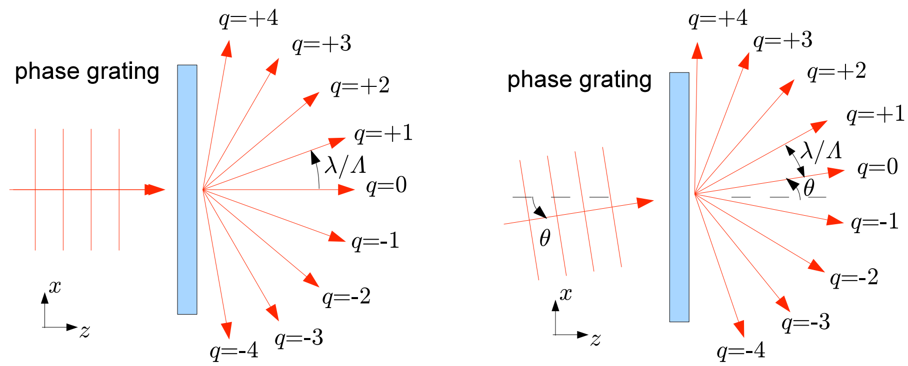
Since $\exp\left\{i\frac{2\pi}{\lambda}\left(\theta + \frac{q\lambda}{\Lambda}\right)x\right\}$ represents a tilted plane wave whose propagation angle is $\theta + q\lambda/\Lambda$, eq. (3) implies that the transmitted field just behind the grating is consisted of a infinite number of plane waves, where q denotes diffraction order and the amplitude of the diffraction order q is $J_q(m/2)$. The propagation direction of the zero–order is identical as one of the incident tilted plane wave.

1.b) The field behind the grating is identical to eq. (1). When the observation plane is in the far–zone, the Fraunhofer diffraction pattern is

$$g(x', z) = \int g_+(x, z = 0)\exp\left\{-i\frac{2\pi}{\lambda z}(x'x)\right\}\mathrm{d}x. \quad (4)$$

Note that we neglected the scaling factor and phase term because the scaling factor change overall magnitude of diffraction pattern and the phase term does not contribute to intensity. Substituting eq. (2) into (4), we obtain the field distribution of the Fraunhofer diffraction as

$$\begin{aligned} g(x', z) &= \int \left[\sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\exp\left\{i\frac{2\pi}{\lambda}\left(\theta + \frac{q\lambda}{\Lambda}\right)\right\}\right]\exp\left\{-i\frac{2\pi}{\lambda z}(xx')\right\}\mathrm{d}x \\ &= \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\left[\int \exp\left\{i2\pi\left(\frac{q}{\Lambda} + \frac{\theta}{\lambda}\right)x\right\}\exp\left\{-i2\pi\frac{x'}{\lambda z}x\right\}\mathrm{d}x\right] \\ &= \sum_{q=-\infty}^{\infty} J_q\left(\frac{m}{2}\right)\delta\left(\frac{x'}{\lambda z} - \left[\frac{q}{\Lambda} + \frac{\theta}{\lambda}\right]\right). \end{aligned} \quad (5)$$



(a) on–axis plane wave (b) tilted plane wave

Figure 1: The whole diffraction patterns rotate by θ as the incident plane wave rotates

The intensity of the Fraunhofer diffraction pattern is

$$I(x', z) = |g(x', z)|^2 = \sum_{q=-\infty}^{\infty} J_q^2\left(\frac{m}{2}\right)\delta\left(\frac{x'}{\lambda z} - \left[\frac{q}{\Lambda} + \frac{\theta}{\lambda}\right]\right). \tag{6}$$

In the far–region, we should observe a infinite number of diffraction orders. The intensity of the diffraction order is proportional to $J_q^2(m/2)$ and the offset between two neighboring diffraction orders is $(\lambda z)/\Lambda$. The zeroth order is located at $x' = z\theta$.

1.c) In both cases (Fresnel and Fraunhofer diffraction), the diffraction patterns of the grating probed by a on-axis and tilted plane waves are identical except the angular shift by the incident angle θ, as shown in Fig. 1.

Problem 2: Grating spherical wave illumination

2.a) Using the same approach as in Prob. 1, we obtain

$$g_+(x,y,z=0) = g_t(x,y)g_-(x,y,z=0) = \frac{1}{2}\left[1+m\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}\right\}. \tag{7}$$

2.b) Since both the cosine term and exponential terms in eq. (8) vary with x, we use following relation to understand eq. (8);

$$1+m\cos\left(2\pi\frac{x}{\Lambda}\right) = 1+\frac{m}{2}\left(\exp\left\{i2\pi\frac{x}{\Lambda}\right\}+\exp\left\{-i2\pi\frac{x}{\Lambda}\right\}\right). \tag{8}$$

Hence, eq. (8) can be rewritten as superposition of three spherical waves;

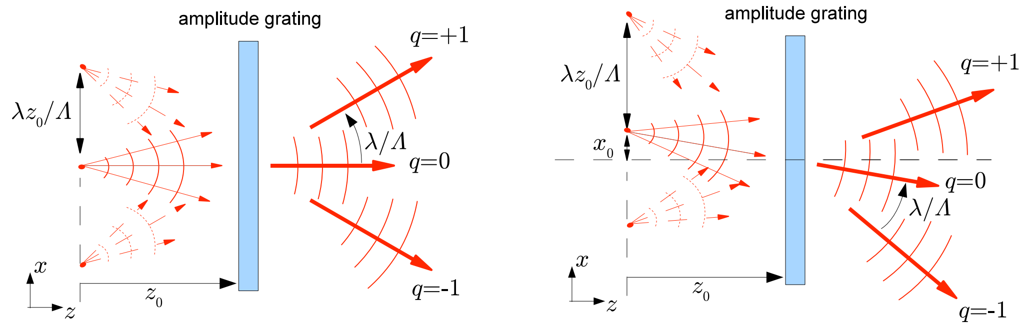
$$\begin{aligned}
g_+(x,y,z=0) &= \frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\left[\frac{1}{2}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}\right\}+\right.\\
&\quad\left.\frac{m}{4}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}+i2\pi\frac{x}{\Lambda}\right\}+\frac{m}{4}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}-i2\pi\frac{x}{\Lambda}\right\}\right]\\
&= \frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\left[\frac{1}{2}\exp\left\{i\pi\frac{x^2+y^2}{\lambda z_0}\right\}+\frac{m}{4}\exp\left\{i\pi\frac{(x+\lambda z_0/\Lambda)^2+y^2}{\lambda z_0}\right\}\exp\left\{-i\pi\frac{\lambda z_0}{\Lambda^2}\right\}+\right.\\
&\quad\left.\frac{m}{4}\exp\left\{i\pi\frac{(x-\lambda z_0/\Lambda)^2+y^2}{\lambda z_0}\right\}\exp\left\{-i\pi\frac{\lambda z_0}{\Lambda^2}\right\}\right].
\end{aligned} \tag{9}$$

2.c) Figure 2(a) conceptually shows the diffraction pattern expressed in eq. (10). The first exponential term represents the zero–order diffraction, which is identical to the incident spherical wave originated at $(x=0, y=0, z=-z_0)$ except amplitude attenuation. The second and third exponential terms indicate two spherical waves originated at $(\pm\lambda z_0/\Lambda, 0, -z_0)$ with additional phase factor of $e^{-i\pi\lambda z_0/\Lambda^2}$, which is independent on x and y.

2.d) If the illumination is a spherical wave emitted at $(x_0, 0, -z_0)$ as shown in Fig. 2(b), then we expect that the origins of the three spherical waves will be shifted by x_0; i.e., the three origins are $(x_0, 0, -z_0)$, $(x_0-\lambda z_0/\Lambda, 0, -z_0)$, and $(x_0+\lambda z_0/\Lambda, 0, -z_0)$ if the paraxial approximation holds.

More rigorously, the Fresnel diffraction pattern is computed as

$$g_+(x,z=0) = \frac{1}{2}\left[1+m\cos\left(2\pi\frac{x}{\Lambda}\right)\right]\frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\exp\left\{i\pi\frac{(x-x_0)^2+y^2}{\lambda z_0}\right\}, \tag{10}$$



(a) on–axis spherical wave

(b) off–axis spherical wave

Figure 2: The diffraction patterns rotate in the same fashion as the incident spherical wave rotates

and using the same expansion we eventually obtain

$$
\begin{aligned}
g_+(x, y, z = 0) = \frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\Bigg[\frac{1}{2}\exp\left\{i\pi\frac{(x-x_0)^2+y^2}{\lambda z_0}\right\} + \\
\frac{m}{4}\exp\left\{i\pi\frac{(x-x_0)^2+y^2}{\lambda z_0} + i2\pi\frac{x}{\Lambda}\right\} + \frac{m}{4}\exp\left\{i\pi\frac{(x-x_0)^2+y^2}{\lambda z_0} - i2\pi\frac{x}{\Lambda}\right\}\Bigg] \\
= \frac{e^{i2\pi\frac{z_0}{\lambda}}}{i\lambda z_0}\Bigg[\frac{1}{2}\exp\left\{i\pi\frac{(x-x_0)^2+y^2}{\lambda z_0}\right\} + \\
\frac{m}{4}\exp\left\{i\pi\frac{(x-x_0+\lambda z_0/\Lambda)^2+y^2}{\lambda z_0}\right\}\exp\left\{-i\pi\left(-\frac{2x_0}{\Lambda}+\frac{\lambda z_0}{\Lambda^2}\right)\right\} + \\
\frac{m}{4}\exp\left\{i\pi\frac{(x-x_0-\lambda z_0/\Lambda)^2+y^2}{\lambda z_0}\right\}\exp\left\{-i\pi\left(\frac{2x_0}{\Lambda}+\frac{\lambda z_0}{\Lambda^2}\right)\right\}\Bigg]. \quad (11)
\end{aligned}
$$

As expected, the diffraction pattern is consisted of three spherical waves originated at $(x_0, 0, -z_0)$, $(x_0 - \lambda z_0/\Lambda, 0, -z_0)$, and $(x_0 + \lambda z_0/\Lambda, 0, -z_0)$, respectively.

3. (a) The Fourier series coefficients of a periodic function $t(x)$ are given by:

$$a_0 = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} t(x')dx'$$

$$a_n = \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} t(x')\cos\left(\frac{n\pi x'}{L/2}\right)dx'$$

$$b_n = \frac{2}{L}\int_{-L}^{L} t(x')\sin\left(\frac{n\pi x'}{L/2}\right)dx'$$

where L is the period of $t(x)$. The function $t(x)$ can then be written as an infinite sum:

$$t(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L/2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L/2}\right)$$

For the given function,

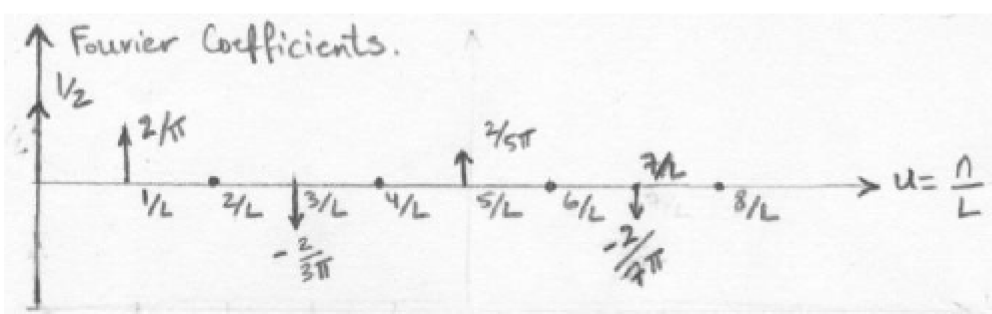
$$a_0 = \frac{1}{L}\int_{-\frac{L}{4}}^{\frac{L}{4}} dx' = \frac{1}{2}$$

$$a_n = \frac{2}{L}\int_{-\frac{L}{4}}^{\frac{L}{4}} \cos\left(\frac{2\pi n x'}{L}\right)dx' = \frac{\cancel{2}}{\cancel{L}}\cdot\frac{\cancel{L}}{\cancel{2}\pi n}\sin\left(\frac{2\pi n x'}{L}\right)\Bigg|_{-\frac{L}{4}}^{\frac{L}{4}} = \frac{2}{\pi n}\sin\left(\frac{\pi n}{2}\right)$$

$$b_n = \frac{2}{L}\int_{-\frac{L}{4}}^{\frac{L}{4}} \sin\left(\frac{2\pi n x'}{L}\right)dx' = \frac{\cancel{2}}{\cancel{L}}\cdot\frac{\cancel{L}}{\cancel{2}\pi n}\underbrace{\left[-\cos\left(\frac{2\pi n x'}{L}\right)\Bigg|_{-\frac{L}{4}}^{\frac{L}{4}}\right]}_{0} = 0$$

$$\therefore a_0 = \frac{1}{2}, \quad b_n = 0, \quad a_n = \frac{\sin\left(\frac{\pi n}{2}\right)}{\frac{\pi n}{2}} \text{ where } n = 1, 2, 3, \ldots$$

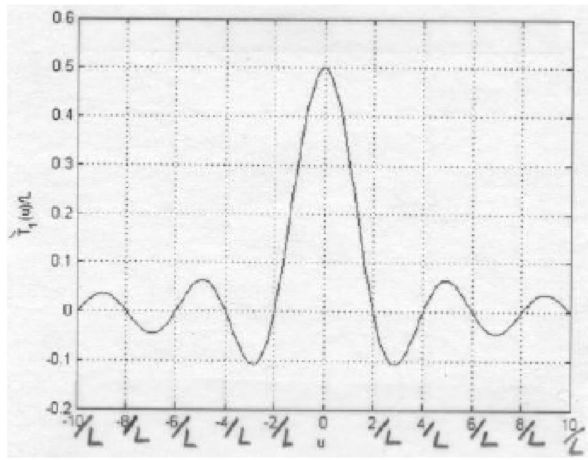
Note that when n is even, $a_n = 0$, when $n = 1 + 4m, a_n = +1$, and when $n = 3 + 4m, a_n = -1$, where m is a positive integer.



(b) A single boxcar is given by

$$t_1(x) = \text{rect}\left(\frac{2x}{L}\right)$$

$$T_1(u) = \mathcal{F}(t_1(x)) = \frac{L}{2}\text{sinc}\left(\frac{L}{2}u\right)$$



(c) An infinite array of boxcars of width $\frac{L}{2}$ with a spacing of $\frac{L}{2}$ between them can be expressed as a convolution of a comb() function and a rect() function:

$$t_2(x) = \text{rect}\left(\frac{2x}{L}\right) \otimes \text{comb}\left(\frac{x}{L}\right)$$

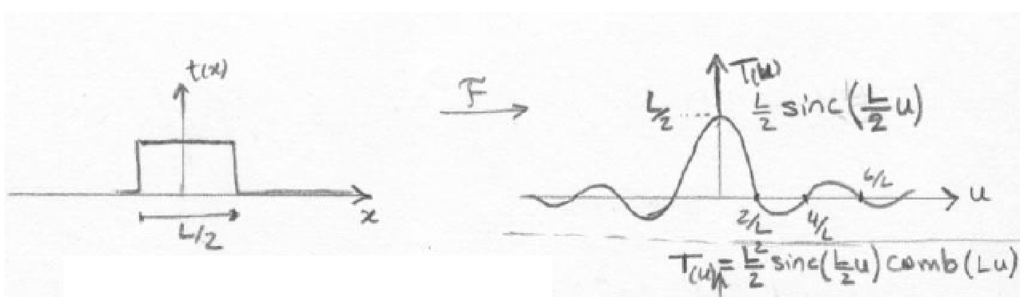
A truncated centered portion containing N boxcars is then given by

$$t(x) = t_2(x) \cdot \text{rect}\left(\frac{x}{NL}\right) = \left[\text{rect}\left(\frac{2x}{L}\right) \otimes \text{comb}\left(\frac{x}{L}\right)\right] \cdot \text{rect}\left(\frac{x}{NL}\right)$$

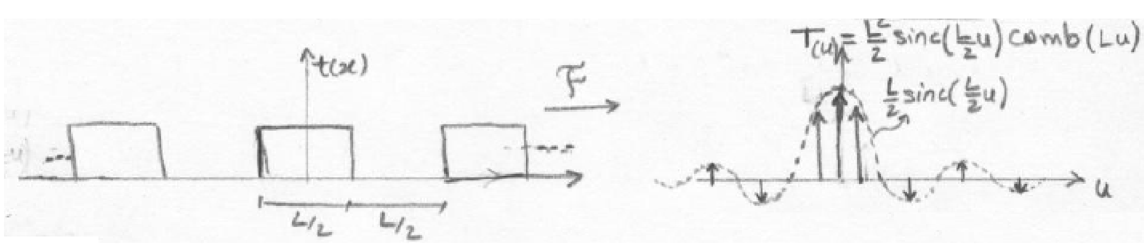
The Fourier transform of $t(x)$ becomes

$$T(u) = T_2(u) \otimes (\text{NL})\text{sinc}(\text{NL}u) = \left[\frac{L^2}{2}\text{sinc}\left(\frac{L}{2}u\right) \cdot \text{comb}(Lu)\right] \otimes (\text{NL})\text{sinc}(\text{NL}u)$$

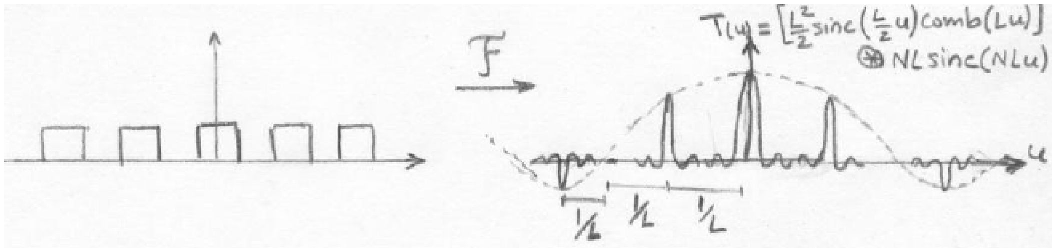
(d) <u>Single box car:</u> $T(u) = \frac{L}{2}\text{sinc}(\frac{L}{2}u)$



<u>Infinite array:</u>

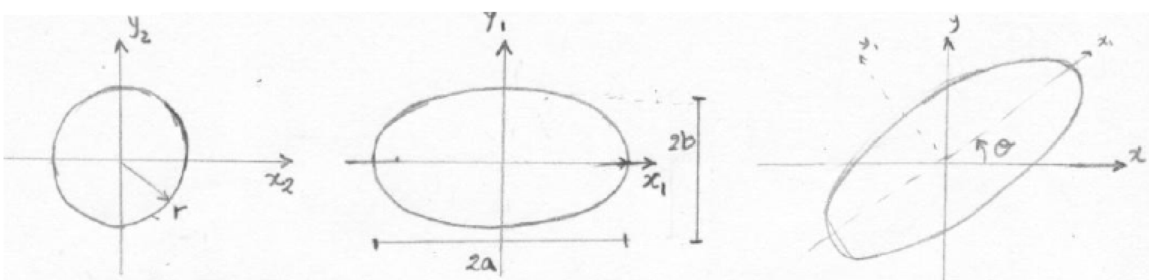


Finite array (N):



- The Fraunhofer diffraction pattern is similar to the Fourier transform of the functions (with a scaling factor $u = x/\lambda L$)
- A single box car creates a sinc() diffraction pattern. Having an infinitely long array would generate a set of $\delta()$ functions, i.e. single dots whose amplitude is modulated by a sinc() envelope profile identical to that generated by one boxcar. The spacing of the $\delta()$'s is the reciprocal of the period of the array.
- A finite array of boxcars generates a set of sinc() functions whose peaks are modulated by another sinc() function and whose spacing is the reciprocal of the period of the boxcar array. Limiting the size of the array is equivalent to imposing a window onto an infinite array. This window spreads the $\delta()$ functions into sinc() functions. The spread of each of these sinc()'s is inversely proportional to the width of the 'window.'

4. Tilted ellipse:



$$\text{Circle: } f_2(x_2, y_2) = \text{circ}\left(\frac{\sqrt{x^2+y^2}}{r}\right)$$

$$\text{Ellipse: } x_1 = \frac{a}{r}x_2; y_1 = \frac{b}{r}y_2$$

$$x_2 = \frac{r}{a}x_1; y_2 = \frac{r}{b}y_1$$

$$\text{Tilted: } x = x_1\cos\theta - y_1\sin\theta$$

$$y = x_1\sin\theta + y_1\cos\theta$$

$$f_1(x_1, y_2) = \text{circ}\left(\frac{\sqrt{(\frac{r}{a}x_1)^2 + (\frac{r}{b}y_1)^2}}{r}\right) = \text{circ}\left(\sqrt{\left(\frac{x_1}{a}\right)^2 + \left(\frac{y_1}{b}\right)^2}\right)$$

$$F_1(u_1, v_1) = \mathcal{F}(f_1) = |ab|\text{jinc}(2\pi\sqrt{(au_1)^2 + (bv_1)^2})$$

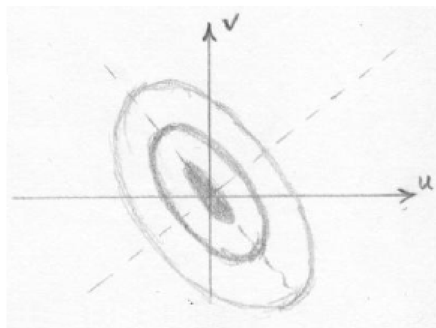
A rotation by θ in the space domain is equivalent to a rotation by θ in the frequency domain; hence,

$$u_1 = u\cos\theta + v\sin\theta, \quad v_1 = -u\sin\theta + v\cos\theta$$

$\therefore$ The Fourier transform of an ellipse tilted by an angle θ is

$$F(u, v) = |ab|\text{jinc}(2\pi\sqrt{a^2(u\cos\theta + v\sin\theta)^2 + b^2(-u\sin\theta + v\cos\theta)^2})$$

(a) Sketch of Fourier transform



(b) The Fraunhofer diffraction pattern is given by

$$F(u, v)\Big|_{(\frac{x'}{\lambda\ell}, \frac{y'}{\lambda\ell})} = |ab|\text{jinc}\left(2\pi\sqrt{a^2(\frac{x'}{\lambda\ell}\cos\theta + \frac{y'}{\lambda\ell}\sin\theta)^2 + b^2(-\frac{x'}{\lambda\ell}\sin\theta + \frac{y'}{\lambda\ell}\cos\theta)^2}\right)$$

Problem 5: Blazed grating In this problem, the given phase profile of the grating is shown in Fig. 3. So it is important to properly construct the complex transparency of the grating.

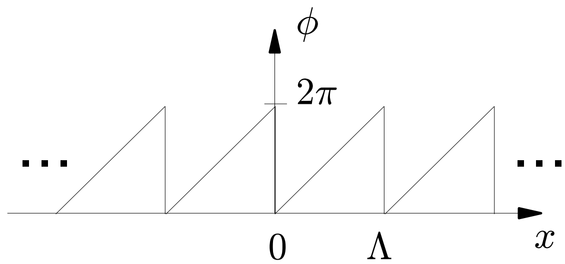


Figure 3: phase profile of the blazed grating

3.a) Since the grating is a periodic function without any truncation, Fourier series can be used and the Fourier coefficient (its square for intensity) represents the diffraction efficiency.

$$\begin{aligned} a_q = \frac{1}{\Lambda}\int_0^\Lambda \exp\left\{i\frac{2\pi}{\Lambda}x\right\}\exp\left\{-i2\pi\frac{q}{\Lambda}x\right\}\mathrm{d}x \\ = \frac{1}{\Lambda}\int_0^\Lambda \exp\left\{i\frac{2\pi}{\Lambda}(1-q)\,x\right\}\mathrm{d}x = \frac{1}{\Lambda}\frac{\exp\{i2\pi(1-q)\}-1}{i\frac{2\pi}{\Lambda}(1-q)} \\ = e^{i\pi(1-q)}\frac{\sin(\pi(1-q)}{\pi(1-q)} = e^{-i\pi q}\mathrm{sinc}\,(q-1)\,, \end{aligned} \quad (12)$$

where q denotes the diffraction order. Hence, the efficiency of the diffraction order q is proportional to

$$I_q \sim |a_q|^2 = \mathrm{sinc}^2(q-1). \quad (13)$$

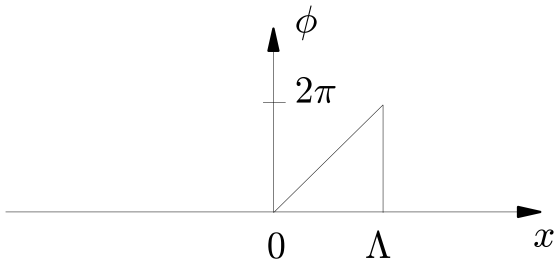
Note that only the first order is survived and other orders are canceled out. The blazed grating produces the same number of diffraction orders as non–blazed gratings; but the blazed grating concentrates more light into a specific order due to the phase profile. (You can imagine that the one period of the grating behaves as a prism to bend light!)

The same conclusion can be obtained with Fourier transform. We first choose one period of the grating whose complex transparency (not phase profile!) as

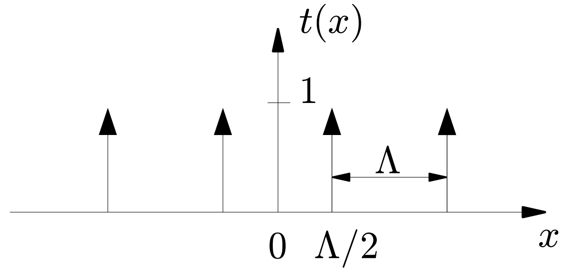
$$t_\Lambda(x) = \mathrm{rect}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right)e^{i2\pi\frac{x}{\Lambda}}, \quad (14)$$

and it is shown in Fig. 4(a). To make the grating periodic, it convolves with a comb function (impulse train) whose period is Λ shown in Fig. 4(b). Note the comb function is shifted by $\Lambda/2$ to correctly represent the given grating. Then, the complex transparency of the grating can be written as

$$t(x) = \left[\mathrm{rect}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right)e^{i2\pi\frac{x}{\Lambda}}\right]\otimes\mathrm{comb}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right). \quad (15)$$



(a) phase of the one period of the grating



(b) desired comb function

Figure 4: The grating can be constructed by the convolution of a triangular phase profile and comb function.

To find the diffraction efficiencies, we compute Fraunhofer diffraction pattern, which is identical to taking Fourier transform of the complex transparency.

$$\begin{aligned}
&\mathfrak{F}\left\{\left[\text{rect}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right)e^{i2\pi\frac{x}{\Lambda}}\right]\otimes\text{comb}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right)\right\}= \\
&\mathfrak{F}\left\{\text{rect}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right)e^{i2\pi\frac{x}{\Lambda}}\right\}\mathfrak{F}\left\{\text{comb}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right)\right\}= \\
&\left\{\left[\Lambda\text{sinc}\left(\Lambda u\right)e^{-i2\pi\frac{\Lambda}{2}u}\right]\otimes\left[\delta\left(u-\frac{1}{\Lambda}\right)\right]\right\}\left\{\Lambda\text{comb}\left(\Lambda u\right)e^{-i2\pi\frac{\Lambda}{2}u}\right\}= \\
&\Lambda^2\left\{\text{sinc}\left(\Lambda\left[u-\frac{1}{\Lambda}\right]\right)e^{-i2\pi\frac{\Lambda}{2}[u-\frac{1}{\Lambda}]}\right\}\text{comb}\left(\Lambda u\right)e^{-i2\pi\frac{\Lambda}{2}u}= \\
&\Lambda^2 e^{-i2\pi\Lambda[u-\frac{1}{2\Lambda}]}\left\{\text{sinc}\left(\Lambda\left[u-\frac{1}{\Lambda}\right]\right)\right\}\text{comb}\left(\Lambda u\right). \quad (16)
\end{aligned}$$

Thus, the intensity of the diffraction pattern is proportional to

$$I(u)\sim\text{sinc}^2\left(\Lambda\left[u-\frac{1}{\Lambda}\right]\right)\text{comb}\left(\Lambda u\right), \quad (17)$$

where the sinc^2 and comb functions are shown in Fig. 4(a) and (b). Fig. 4(c) shows the Fraunhofer diffraction pattern replacing with $x'=u\lambda z$ at an observation plane. Again, the diffraction efficiency of the diffraction order q, which is defined by the comb function, is found to be

$$I_q\sim\text{sinc}^2\left(q-1\right). \quad (18)$$

Note that the lateral shift of the grating does not make any significant change in the intensity of the Fraunhofer diffraction pattern; i.e., if the grating is shifted by $\Lambda/2$ laterally, then the complex transparency would be

$$t(x)=\left[\text{rect}\left(\frac{x}{\Lambda}\right)e^{i2\pi\frac{x-\frac{\Lambda}{2}}{\Lambda}}\right]\otimes\text{comb}\left(\frac{x}{\Lambda}\right). \quad (19)$$

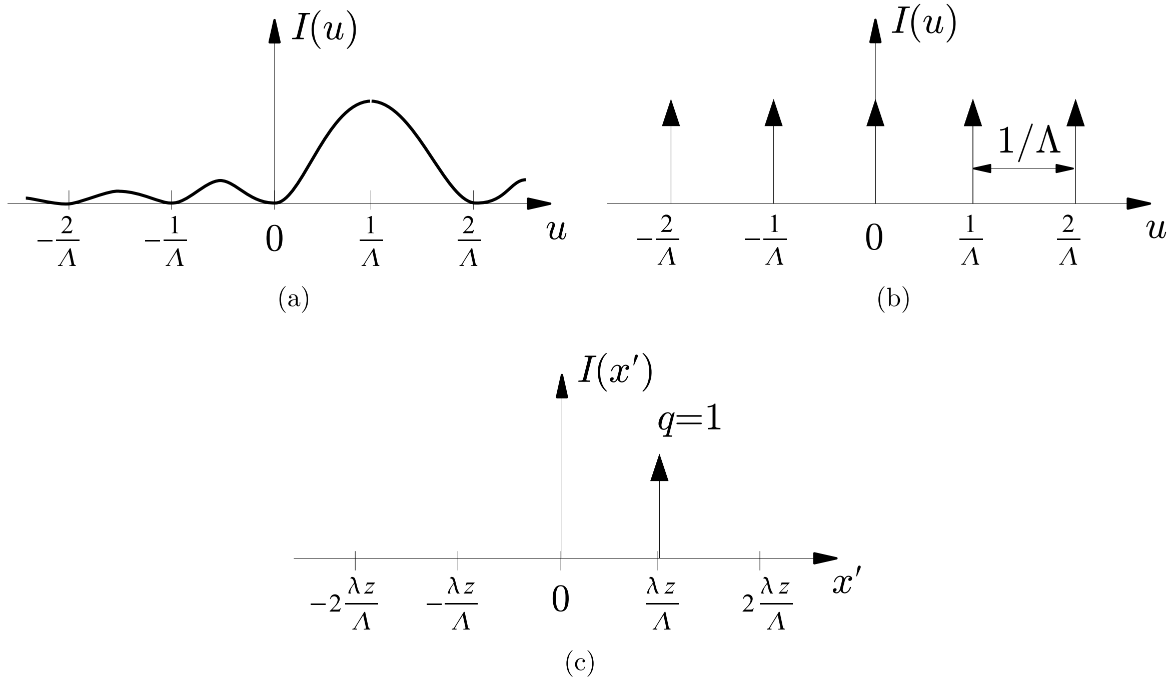


Figure 5: The Fraunhofer diffraction pattern

Using the same procedure, we obtain the same result as

$$\mathfrak{F}\left\{\left[e^{-i\pi}\mathrm{rect}\left(\frac{x}{\Lambda}\right)e^{i2\pi\frac{x}{\Lambda}}\right]\otimes\mathrm{comb}\left(\frac{x}{\Lambda}\right)\right\}=\Lambda^2e^{-i\pi}\mathrm{sinc}\left(\Lambda\left[u-\frac{1}{\Lambda}\right]\right)\mathrm{comb}\left(\Lambda u\right) \quad (20)$$

and

$$I_q\sim\mathrm{sinc}^2\left(q-1\right). \quad (21)$$

3.b) If the phase contrast is reduced from 2π to ϕ_0, then the complex transparency of the new grating is written as

$$t(x)=\left[\mathrm{rect}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right)e^{i\phi_0\frac{x}{\Lambda}}\right]\otimes\mathrm{comb}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right). \quad (22)$$

Using the same procedure, we obtain

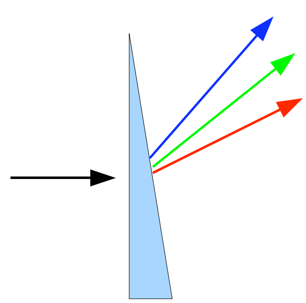
$$\begin{aligned}\mathfrak{F}\left\{t(x)\right\}=\mathfrak{F}\left\{\mathrm{rect}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right)e^{i\phi_0\frac{x}{\Lambda}}\right\}\mathfrak{F}\left\{\mathrm{comb}\left(\frac{x-\frac{\Lambda}{2}}{\Lambda}\right)\right\}=\\ \left\{\left[\Lambda\mathrm{sinc}\left(\Lambda u\right)e^{-i2\pi\frac{\Lambda}{2}u}\right]\otimes\left[\delta\left(u-\frac{\phi_0}{2\pi\Lambda}\right)\right]\right\}\left\{\Lambda\mathrm{comb}\left(\Lambda u\right)e^{-i2\pi\frac{\Lambda}{2}u}\right\}=\\ \Lambda^2\left\{\mathrm{sinc}\left(\Lambda\left[u-\frac{\phi_0}{2\pi\Lambda}\right]\right)e^{-i2\pi\frac{\Lambda}{2}[u-\frac{\phi_0}{2\pi\Lambda}]}\right\}\mathrm{comb}\left(\Lambda u\right)e^{-i2\pi\frac{\Lambda}{2}u}=\\ \Lambda^2e^{-i2\pi\Lambda[u-\frac{\phi_0}{4\pi\Lambda}]}\left\{\mathrm{sinc}\left(\Lambda\left[u-\frac{\phi_0}{2\pi\Lambda}\right]\right)\right\}\mathrm{comb}\left(\Lambda u\right). \quad (23)\end{aligned}$$

Hence, the efficiency of the diffraction order q is proportional to

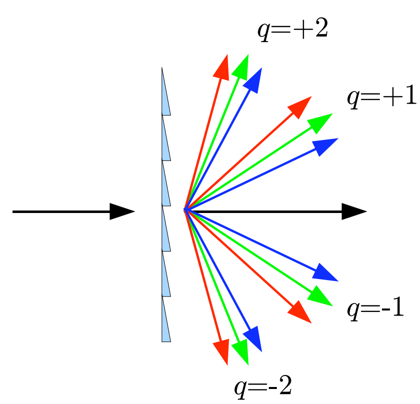
$$I_q \sim \text{sinc}^2\left(q - \frac{\phi_0}{2\pi}\right). \tag{24}$$

Note that now we have an infinite number of diffraction orders, and the efficiencies can be tuned by changing the phase modulation. The Fourier series approach should expect the same result.

3.c) Since a prism exhibits refraction, all incoming light bends based on the Snell's law. The prism does not produce additional diffraction orders. Also the dispersion of the prism is normal dispersion, in which long wavelength bends less than short wavelength, as shown in Fig. 6(a). The blazed grating, even though its shape of the profile is similar to the prism, it exhibits diffraction and produces multiple diffraction orders. The zero–order diffraction keeps propagating along the same direction as the incident wave but no dispersion happens, Other higher orders, whose efficiency is dependent on the modulation of the phase (in this problem, ϕ_0), are p in the same way of the incident light except the amplitude attenuation. This phase grating produces an infinite numbers of diffraction, and the efficiency of the diffraction order depends on the modulation of the phase (in here, ϕ_0). The dispersion of the grating is anormal, in which longer wavelength diffracts more than short wavelength. Another way to understand is that since $u = x/(\lambda z)$. Thus, although spatial frequencies are same, they appear at different location due to $x = \lambda z u$, where longer wavelength length deviates more.



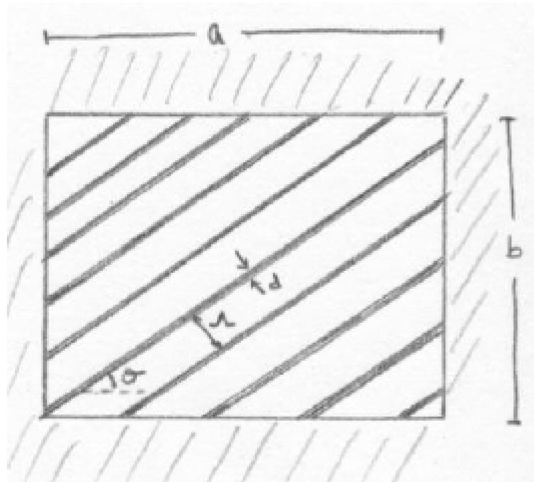
(a) Prism



(b) Blazed grating

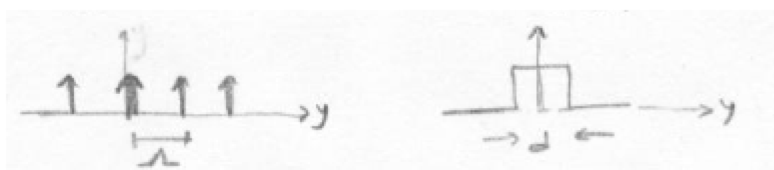
Figure 6: Comparison of a prism and blazed grating

6. The grating can be expressed as a convolution of a rect() function and a comb() function, where the comb() is an infinite sum of equally spaced $\delta()$ functions.



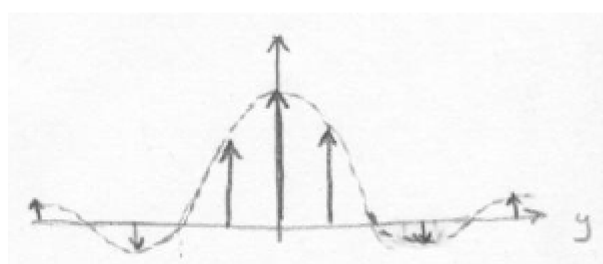
The truncated grating is the product of a 2D rect() function (the window) and the infinite grating.

Upright grating: $\displaystyle\sum_{n=+-\infty}^{+\infty} \delta(y - n\Lambda) \otimes \text{rect}(y/d)$



From problem # 2, we know that this convolution results in:

$$\sum_{n=-\infty}^{\infty} \underbrace{\frac{2\sin(n\frac{2\pi}{\Lambda}\frac{d}{2})}{n}}_{\text{sinc envelope}} \cdot \delta(u - n\tfrac{2\pi}{\Lambda}, v)$$



A rotation by 60° in space results in a rotation by 60° in the frequency domain:

$$u' = u\cos 60^\circ + v \sin 60^\circ, v' = -u \sin 60^\circ + v\cos 60^\circ$$

$$\sum_{n=-\infty}^{\infty} \frac{2\sin(n\frac{2\pi}{\Lambda}\frac{d}{2})}{n} \cdot \delta\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}v - n\frac{2\pi}{\Lambda}, -\frac{\sqrt{3}}{2}u + \frac{1}{2}v\right) = F_1(u, v)$$

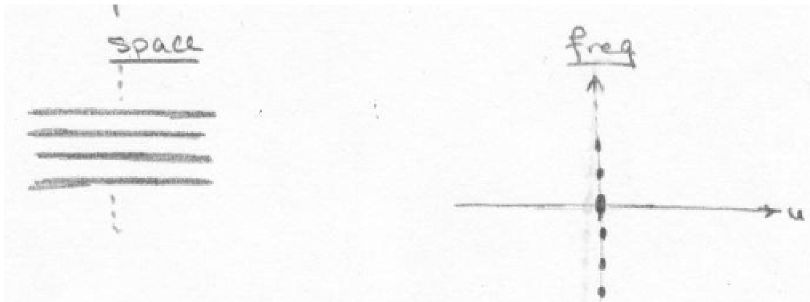
The Fourier transform of the truncated grating is the convolution of F_1 with the Fourier transform of the rectangular window:

$$F(u, v) = F_1(u, v) \otimes |ab|\text{sinc}(au)\text{sinc}(bv)$$

The Fraunhofer diffraction pattern is $\left.\frac{e^{i2\pi\frac{\ell}{\lambda}}}{i\lambda\ell}[F(u,v)]\right|_{(\frac{x}{\lambda\ell},\,\frac{y}{\lambda\ell})}$

Sketches of Fourier Transforms

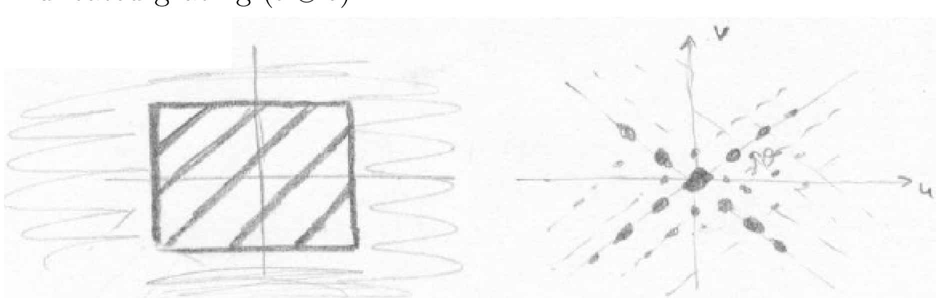
(a) Infinite grating



(b) Rotated infinite grating

(c) Rectangular aperture

(d) Truncated grating ($b \otimes c$)



2.71 / 2.710 Optics
Spring 2009