MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.71 Optics **Spring '09**

Solutions to Problem Set #4

Due Wednesday, April 1, 2009

Problem 1: Particle in quadratic and linear potential

a) Consider a particle whose Hamiltonian is given by,

$$H(q_x, q_z; p_x, p_z) = \frac{p_x^2 + p_z^2}{2m} + \frac{1}{2}kq_x^2 + mgq_x, \tag{1}$$

where m is the particle mass, k is the spring constant, and g is a constant with the units of accelerations. We begin by writing the set of Hamiltonian equations in the same way as done in the class Lecture 9, p. 22,

$$\frac{dq_x}{dt} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}, \tag{2}$$

$$\frac{dq_z}{dt} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}, \tag{3}$$

$$\frac{dp_x}{dt} = -\frac{\partial H}{\partial q_x} = -kq_x - mg, \tag{4}$$

$$\frac{dp_z}{dt} = -\frac{\partial H}{\partial q_z} = 0. \tag{5}$$

b) As indicated by equation 5, the axial momentum is conserved, $p_z(t) = p_z(0)$ =constant, and we solve for $q_z(t)$ from equation 3,

$$\begin{aligned} \frac{dq_z}{dt} &= \frac{p_z(0)}{m} \qquad (6) \\ \Rightarrow \quad q_z(t) &= q_z(0) + \frac{p_z(0)}{m}t. \end{aligned}$$

To find the second-order differential equation for the lateral position, we take the time derivative on both sides of equation 2,

$$\begin{aligned} \frac{d^2q_x}{dt^2} &= \frac{1}{m}\frac{dp_x}{dt} \qquad (7) \\ &= -\frac{1}{m}\left(kq_x + mg\right) \\ \Rightarrow \quad & \frac{d^2q_x}{dt^2} + \frac{k}{m}q_x = -g. \end{aligned}$$

We now solve equation 7 with the initial conditions, $q_x(t=0)=q_0$, $q_z(t=0)=0$, $p_x(t=0)=p_0$, and $p_z(t=0)=p_{z0}$,

$$\Rightarrow q_x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right) + C, \tag{8}$$

where C is the particular solution. Substituting,

$$\begin{aligned} \frac{k}{m}C &= -g \\ &\Rightarrow C = -\frac{gm}{k}. \end{aligned} \tag{9}$$

To find the constants A and B we use the initial conditions,

$$\begin{aligned} q_0 &= A - \frac{gm}{k} \\ &\Rightarrow A = q_0 + \frac{gm}{k}, \end{aligned} \tag{10}$$

$$\begin{aligned} p_0 &= m\frac{dq_x(t)}{dt} = B\sqrt{km} \\ &\Rightarrow B = \frac{p_0}{\sqrt{km}}. \end{aligned} \tag{11}$$

The final solution is,

$$\begin{aligned} q_x(t) &= \left(q_0 + \frac{gm}{k}\right)\cos\left(\sqrt{\frac{k}{m}}t\right) + \frac{p_0}{\sqrt{km}}\sin\left(\sqrt{\frac{k}{m}}t\right) - \frac{gm}{k}, \\ p_x(t) &= -\left(\sqrt{\frac{k}{m}}q_0 + \sqrt{\frac{m}{k}}g\right)\sin\left(\sqrt{\frac{k}{m}}t\right) + p_0\cos\left(\sqrt{\frac{k}{m}}t\right). \end{aligned} \tag{12}$$

c) Figure 1 shows an example of a physical system that conforms to this model consisting of a vertical spring on a cart with frictionless horizontal motion.

d) To verify if the Hamiltonian is conserved, we take its time derivative,

$$\begin{aligned} \frac{dH}{dt} &= \frac{1}{2m}\left(2p_x\frac{\partial p_x}{\partial t} + 2p_z\frac{\partial p_z}{\partial t}\right) + kq_x\frac{\partial q_x}{\partial t} + mg\frac{\partial q_x}{\partial t} \\ &= \frac{1}{m}\left(p_x\left(-\frac{\partial H}{\partial q_x}\right) + p_z\left(-\frac{\partial H}{\partial q_z}\right)\right) + (kq_x + mg)\frac{\partial H}{\partial p_x} \\ &= -\frac{p_x}{m}(kq_x + mg) + (kq_x + mg)\frac{p_x}{m} = 0. \end{aligned} \tag{13}$$

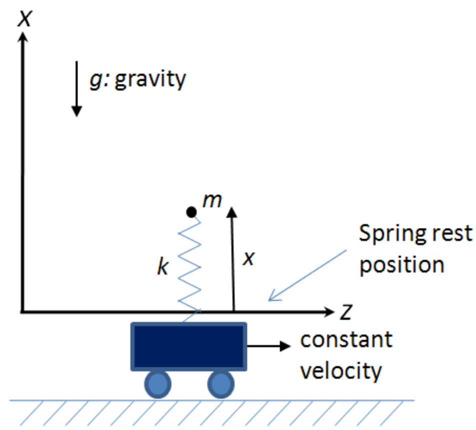

Figure 1: Mechanical system of problem 1.

Problem 2: Quadratic GRIN

a) Consider the quadratic GRIN profile that is most often implemented in practice,

$$n(x) = n_0^2 - \frac{\alpha}{2} q_x^2. \tag{14}$$

The geometry of this problem is shown in Figure 2. The 6 × 6 set of Hamiltonian equations that we saw in class can be simplified to a 4 × 4 set of ordinary differential equations known as *Screen Hamiltonian equations*. The Screen Hamiltonian equations describe the evolution of the intersection of the ray path with the screens, that are perpendicular to the optical axis, as z advances. As shown in Figure 2, different points in the ray trajectory $(s_1, s_2, s_3, \cdots)$ have been projected to their corresponding axial coordinate $(z_1, z_2, z_3, \cdots)$ changing the parameterization of the ray from $[\mathbf{q}(s), \mathbf{p}(s)]$ to $[\mathbf{q}(z), \mathbf{p}(z)]$.The Screen Hamiltonian equations are given by,

$$\begin{aligned} \frac{dq_x}{dz} &= \frac{\partial h}{\partial p_x}, & \frac{dp_x}{dz} &= -\frac{\partial h}{\partial q_x}, \\ \frac{dq_y}{dz} &= \frac{\partial h}{\partial p_y}, & \frac{dp_y}{dz} &= -\frac{\partial h}{\partial q_y}, \end{aligned}$$

where,

$$h(q_x, q_y, z; p_x, p_y) = -\sqrt{n^2(\mathbf{q}) - \left(p_x^2 + p_y^2\right)},$$

is called the *Screen Hamiltonian*. For the case of the quadratic GRIN profile of equation 14, the Screen Hamiltonian becomes,

$$\begin{aligned} h &= -\sqrt{\left(n_0^2 - \frac{\alpha}{2} q_x^2\right)^2 - p_x} \\ &= -\frac{1}{2}\sqrt{4n_0^4 - 4\alpha n_0^2 q_x^2 + \alpha^2 q_x^4 - 4p_x^4}. \end{aligned} \tag{15}$$

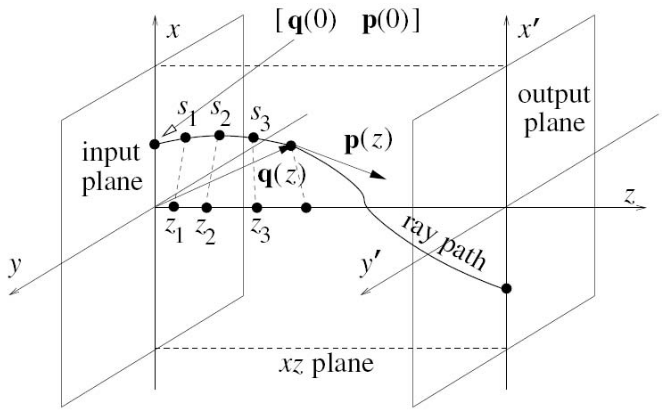

Figure 2: Screen Hamiltonian.

Figure 3 shows the ray position, $q_x(z)$, as a function of z for a case where the index is modulated elliptically,

$$n(x) = \sqrt{n_o^2 - \kappa^2 q_x^2}, \tag{16}$$

with the following parameters: $p_0 = 0$, $n_0 = 1.5$, $k = 0.1$, $q_0 = [-10, 10]$, and $0 \leq z \leq 25$.

b) As you can see from Figure 3, the elliptical GRIN lens doesn't focus the incident parallel ray bundle satisfactory as it suffers a large degree of spherical aberration.

c) Now we consider the quadratic GRIN lens of equation 14. To modify the file, we take the derivative of the Screen Hamiltonian respect to the lateral component of the position vector,

$$\begin{aligned} \frac{\partial h}{\partial q_x} &= -\frac{1}{4}\frac{-8n_0^2\alpha q_x + 4\alpha^2 q_x^3}{\sqrt{4n_0^4 - 4\alpha n_0^2 q_x^2 + \alpha^2 q_x^4 - 4p_x^4}} \\ &= \frac{2n_0^2\alpha q_x - \alpha^2 q_x^3}{2h}. \end{aligned} \tag{17}$$

We use equation 17 to edit the `sgradh_quadratic_hw.m` function. Figure 4 shows a comparison of the ray tracing of the quadratic GRIN lens (blue-solid line) and the elliptical GRIN lens (red-dashed line). As shown in the figure, the quadratic GRIN lens also suffers from spherical aberrations that affect the focusing quality.

d) As indicated by equation 14, the refractive index only varies as function of x so that $\partial n/\partial z = 0$; therefore, the Screen Hamiltonian is conserved.

Problem 3: Complex arithmetic

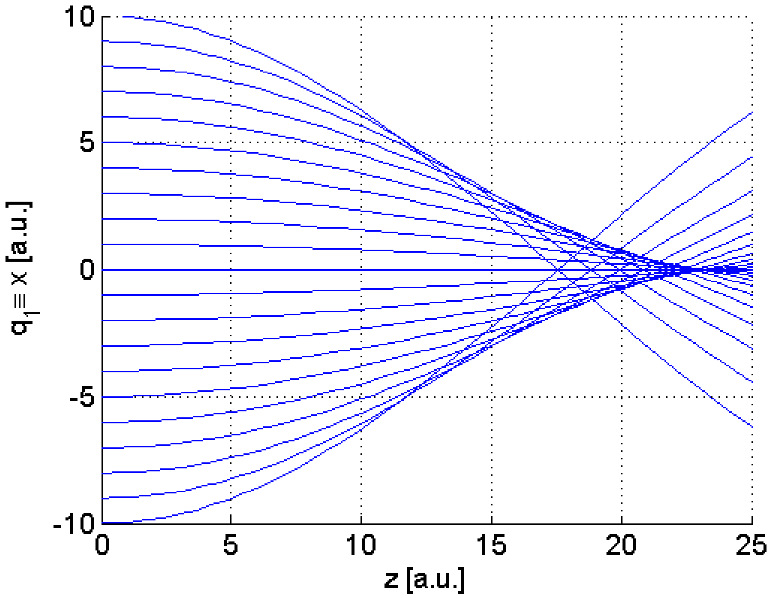

Figure 3: Elliptical GRIN lens.

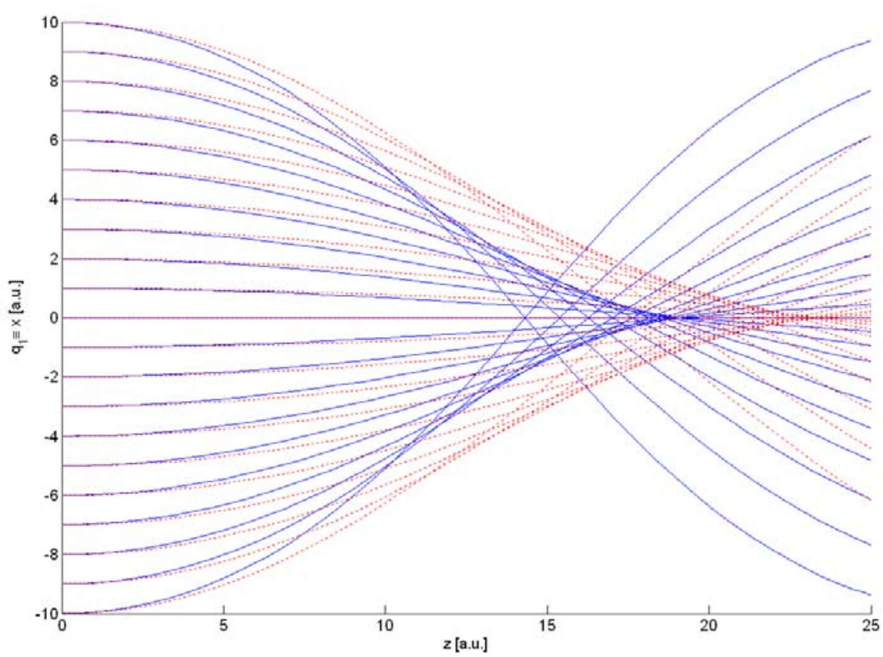

Figure 4: Comparison of quadratic and elliptical GRIN lenses.

a) The goal of this problem is to remind you of some basic complex arithmetic. Let $z_1 = 3 + i4$, $z_2 = 1 - i$, $z_3 = 5e^{i\frac{\pi}{3}}$, and $z_4 = 5e^{i\frac{4\pi}{3}}$. The magnitude and angle of a complex number is given by,

$$\begin{aligned} |z| &= \sqrt{r+i}, \\ \angle z &= \arctan 2\left(\frac{i}{r}\right), \end{aligned} \tag{18}$$

where r is the real part and i is the imaginary part of the complex number z and $\arctan 2$ is the four quadrant tangent function. A complex number written in polar form can be expanded as,

$$\begin{aligned} z &= |z|\, e^{\pm i\theta} \\ &= |z| \cos\theta \pm i \sin\theta, \end{aligned} \tag{19}$$

where $\theta = \angle z$. For the complex numbers given in this problem: $|z_1| = 5$, $|z_2| = 1.4142$, $|z_3| = |z_4| = 5$, $\angle z_1 = 0.9273$, $\angle z_2 = -0.7854$, $\angle z_3 = \pi/3$, and $\angle z_4 = 4\pi/3$.

b) $\angle - z_1 = -2.2143$, $\angle z_2^* = 0.7854$, $\angle - z_3 = -2\pi/3$, and $\angle z_4^* = -4\pi/3$.

c) $z_1 + z_2 = 4 + i3$, $z_1^* + z_2 = 4 + i5$, $z_3 + z_4 = 0$, and $z_1 - z_4^* = 5.5 - i0.33$.

d) $|z_1 z_2| = 7.07$, $|z_3 z_4| = 25$, $|z_3/z_4| = 1$, $\left|\sqrt{z_3}\right| = \sqrt{5}$, $\angle z_1 z_2 = 0.1419$, $\angle z_3 z_4 = -\pi/3$, $\angle z_3/z_4 = -\pi$,and $\angle \sqrt{z_3} = \pi/6$.

e) $z_1 + e^{i\pi} = 2 + i4$, $z_2 e^{i\frac{\pi}{2}} = 1 + i$, $z_3 e^{i\pi} = 5e^{i\frac{4\pi}{3}}$, and $\sqrt{z_4 e^{-i\pi}} = \sqrt{5} e^{i\frac{\pi}{6}}$.

Problem 4: Plane waves and phasor representations

a) We begin by writing the general scalar form of a propagating plane wave in a phasor representation,

$$f(x, y, z, t) = A e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\omega t}, \tag{20}$$

where $\mathbf{r}$ is the position vector, $\mathbf{k}$ is the wave vector with magnitude $|\mathbf{k}| = 2\pi/\lambda$, ω is the angular frequency and A is the amplitude of the wave. For a plane wave propagating at an angle of 30° relative to the $\hat{\mathbf{z}}$ axis on the xz-plane, the wave vector becomes,

$$k = \frac{2\pi}{\lambda} \begin{bmatrix} \sin 30^\circ \\ 0 \\ \cos 30^\circ \end{bmatrix}. \tag{21}$$

For a wavelength $\lambda = 1\mu$m, the wave number is, $k = |\mathbf{k}| = 6.28 \times 10^6 \mathrm{m}^{-1}$. The angular frequency is related to the wavelength by means of the dispersion relation,

$$\begin{aligned} c &= \lambda\nu = \lambda 2\pi\omega \\ \Rightarrow \ \omega &= \frac{c}{\lambda 2\pi} = 4.77 \times 10^{13} \mathrm{rad} \cdot \mathrm{sec}^{-1}. \end{aligned} \tag{22}$$

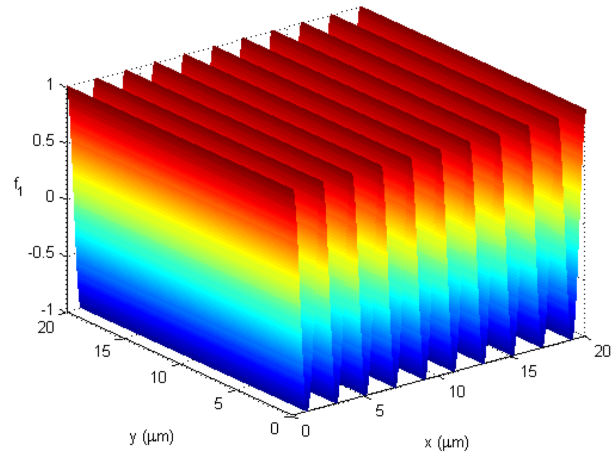

Figure 5: Plane wave propagating at 30° in the xz-plane.

The phasor representation of the wave is,

$$f_1(x, y, z, t) = A \exp\left[ik\left(\sin 30^\circ x + \cos 30^\circ z\right)\right] \exp(-i\omega t), \tag{23}$$

and the space-time representation is,

$$f_1(x, y, z, t) = A \cos[k\left(\sin 30^\circ x + \cos 30^\circ z\right) - \omega t]. \tag{24}$$

b) Similar to part (a), the phasor representation of a plane wave propagating at an angle of 60° relative to the optical axis on the yz-plane is,

$$f_2(x, y, z, t) = A \exp\left[ik\left(\sin 60^\circ y + \cos 60^\circ z\right)\right] \exp(-i\omega t), \tag{25}$$

and the space-time representation is,

$$f_2(x, y, z, t) = A \cos[k\left(\sin 60^\circ y + \cos 60^\circ z\right) - \omega t]. \tag{26}$$

c) Figures 5 and 6 show the waves for $f_1(x, y, z = 0, t = 0)$ and $f_2(x, y, z = 0, t = 0)$.

d) The plane $z = 0$ is illuminated by the superposition of the two waves, f_1 and f_2, and we are interested in plotting the evolution of the resulting wave received at points A, B, C, D, E,

$$(0, 0, 0), \left(\frac{1}{4}, -\frac{1}{4\sqrt{3}}, 0\right), \left(\frac{1}{2}, -\frac{1}{2\sqrt{3}}, 0\right), \left(\frac{3}{4}, -\frac{3}{4\sqrt{3}}, 0\right), \left(1, -\frac{1}{2\sqrt{3}}, 0\right).$$

The evolution of the resulting wave is shown in Figure 7. As shown in this figure, at

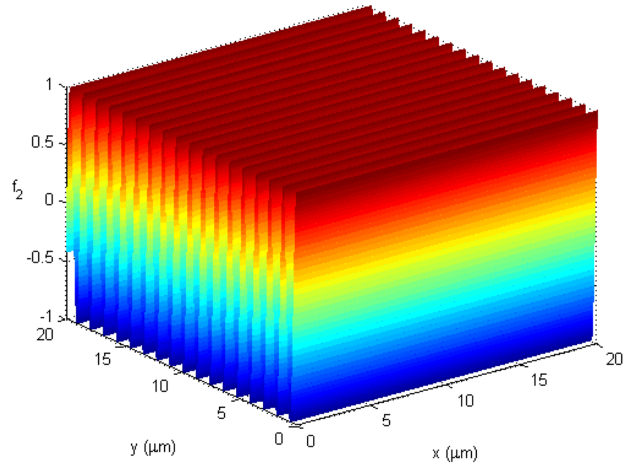

Figure 6: Plane wave propagating at 60° in the xz-plane.

point A the waves f_1 and f_2 are *in phase* so they interfere constructively. In contrast, at point B, the waves are *out-of-phase* and they interfere destructively. An interference pattern is produced at the plane $z = 0$ as a result of the superposition of both waves.

Problem 5: Wave superposition

a) Consider the following two waves,

$$\begin{aligned} f_1(x,z,t) &= 5\cos\left(\frac{2\pi}{17}\left[z + \frac{x^2}{2z}\right] - 2\pi 10t\right), \\ f_2(x,z,t) &= 5\cos\left(\frac{2\pi}{17}\left[z + \frac{(x-5)^2}{2z}\right] - 2\pi 10t + \frac{\pi}{3}\right). \end{aligned} \tag{27}$$

As described in the class (lecture 13, p. 6), the waves of equation 27 are paraxial approximations of spherical waves. For the case of f_1, the originating point source is centered at $(0, 0)$, and the additional parameters are: $A = 5$, $\lambda = 17$, $\nu = 10$. The second wave, f_2, shares the same parameters as f_1; however, the originating point souce is shifted at $x_s = 5$ and the wave is phase shifted by $\phi = \pi/3$.

b) The phase velocity is given by $v_p = \omega/k$. For the two waves of equation 27 their corresponding phase velocities are,

$$v_{p1} = v_{p2} = \lambda\nu = 170. \tag{28}$$

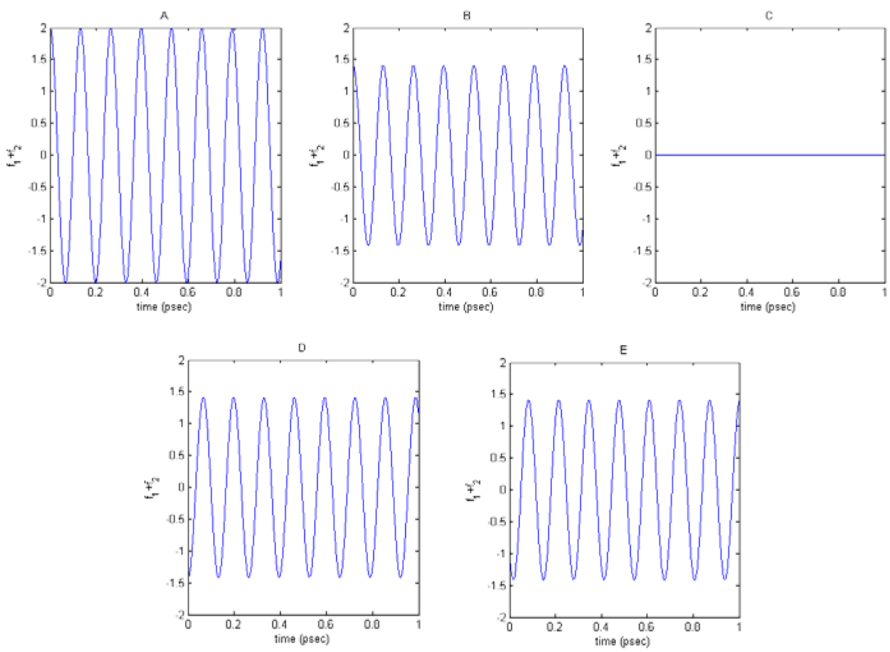

Figure 7: Wave superposition.

c) Now we are interested in computing the coherent superposition of the two waves,

$$\begin{aligned}
f(x,z,t) &= f_1(x,z,t) + f_2(x,z,t) \\
&= 5[\cos\left(\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right] - 2\pi 10t\right) \\
&\quad + \cos\left(\frac{2\pi}{17}\left[z+\frac{(x-5)^2}{2z}\right] - 2\pi 10t + \frac{\pi}{3}\right)] \\
&= 5\left[\cos\left(\phi_1\right) + \cos\left(\phi_2\right)\right] \\
&= 10\left[\cos\left(\frac{\phi_1+\phi_2}{2}\right)\cos\left(\frac{\phi_1-\phi_2}{2}\right)\right] \\
&= 10[\cos\left(\frac{12\pi z^2 + 6\pi x^2 - 2040\pi t z - 30\pi x + 75\pi + 17\pi z}{102z}\right) \\
&\quad \cdot \cos\left(\frac{30\pi x - 75\pi - 17\pi z}{102z}\right)].
\end{aligned} \tag{29}$$

d) The two waves in phasor notation are,

$$\begin{aligned}
f_{p1}(x,z,t) &= 5\exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right] - i2\pi 10t\right), \\
f_{p2}(x,z,t) &= 5\exp\left(i\frac{2\pi}{17}\left[z+\frac{(x-5)^2}{2z}\right] - i2\pi 10t + i\frac{\pi}{3}\right).
\end{aligned} \tag{30}$$

The coherent superposition of the two waves is,

$$\begin{aligned}
f_p(x,z,t) &= f_{p1}(x,z,t) + f_{p2}(x,z,t) \\
&= 5[\exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right] - i2\pi 10t\right) \\
&\quad + \exp\left(i\frac{2\pi}{17}\left[z+\frac{(x-5)^2}{2z}\right] - i2\pi 10t + i\frac{\pi}{3}\right)] \\
&= 5[\exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right] - i2\pi 10t\right) \\
&\quad + \exp\left(i\frac{2\pi}{17}\left[z+\frac{x^2}{2z}\right] - i2\pi 10t\right)\exp\left(i\frac{2\pi}{17}\left[-\frac{5x}{2z}+\frac{25}{2z}\right] + i\frac{\pi}{3}\right)] \\
&= 5\exp(i\phi_1)[1 + \cos\left(\frac{2\pi}{17}\left[\frac{25-5x}{2z}\right] + \frac{\pi}{3}\right) \\
&\quad + i\sin\left(\frac{2\pi}{17}\left[\frac{25-5x}{2z}\right] + \frac{\pi}{3}\right)] \\
&= 5\left[\cos(\phi_1) + i\sin(\phi_1)\right]\left[1 + \cos(\phi_3) + i\sin(\phi_3)\right].
\end{aligned} \tag{31}$$

If we take the real part of equation 31,

$$\begin{aligned} f(x,z,t) &= \text{Re}\{f_p(x,z,t)\} \qquad (32) \\ &= 5\left[\cos(\phi_1)+\cos(\phi_1)\cos(\phi_3)-\sin(\phi_1)\sin(\phi_3)\right] \\ &= 5\left[\cos(\phi_1)+\cos(\phi_1+\phi_3)\right] \\ &= 5\left[\cos(\phi_1)+\cos(\phi_2)\right], \end{aligned}$$

which is the same as in equation 29.

Problem 6: Dispersive waves

a) Recall frrom the class (Lecture 13, p. 7) that the dispersion relation for a metallic waveguide is,

$$\left(\frac{m\pi}{a}\right)^2 + k^2 = \left(\frac{\omega}{c}\right)^2, \qquad (33)$$

where for this problem $\omega = 1.5 \times 10^{15}$rad/sec, $a = 1\mu$m and $m = 1$ since only one mode is allowed. Solving for k,

$$\begin{aligned} k &= \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2} \qquad (34) \\ &= 3.8898 \times 10^6 \text{m}^{-1} \\ &\Rightarrow \lambda_{wg} = \frac{2\pi}{k} = 1.6153 \times 10^{-6}\text{m}. \end{aligned}$$

Comparing equation 34 with the free space wavelength,

$$\lambda_{fs} = \frac{2\pi c}{\omega} = 1.2566 \times 10^{-6}\text{m}. \qquad (35)$$

The temporal period is,

$$T = \frac{2\pi}{\omega} = 4.188 \text{ f}\sec. \qquad (36)$$

Since in the problem statement we are told that the amplitude of the wave is maximum at a distance 0.4μm inside the waveguide at $t = 0$, that is 0.4μm$\approx \lambda_{wg}/4$, the wave is initially phase advanced by $\pi/2$. Figure 8 shows the evolution of the wave at times 1.05fsec, 2.1fsec and 3.15fsec after the wave is launched.

b) As discussed in part (a), the distance traveled by a point of constant phase on the wavefront after 4.2 fsec (temporal period) equals λ_{wg}.The distance traveled by the same point for a wave propagating in free space equals λ_{fs}.

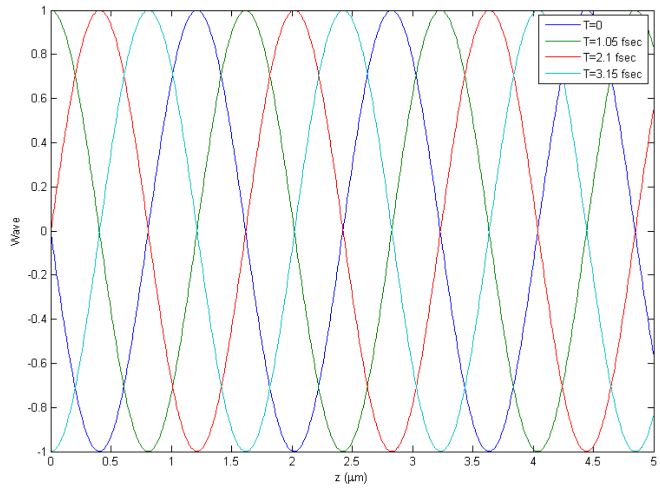

Figure 8: Single mode inside a metallic waveguide.

c) The group velocity is given by,

$$\begin{aligned} v_g &= \frac{\partial \omega}{\partial k} \\ &= \frac{\partial\left(c\sqrt{\left(\frac{m\pi}{a}\right)^2 + k^2}\right)}{\partial k} \\ &= \frac{c^2 k}{c\sqrt{\left(\frac{m\pi}{a}\right)^2 + k^2}} = \frac{c^2 k}{\omega}, \end{aligned} \tag{37}$$

since k is given by equation 34,

$$\begin{aligned} v_g &= \frac{c^2\sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2}}{\omega} \\ &= c\sqrt{1 - \left(\frac{m\pi c}{a\omega}\right)^2}. \end{aligned} \tag{38}$$

Problem 7: Schroedinger's Equation

a) The equation describing the wavepacket associated with a particle in Quantum Mechanics is,

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} = -i\frac{2m}{\hbar}\frac{\partial \Psi}{\partial t}, \tag{39}$$

where, m is the particle mass, $\hbar = h/2\pi$ and h is Planck's constant. Consider a trial

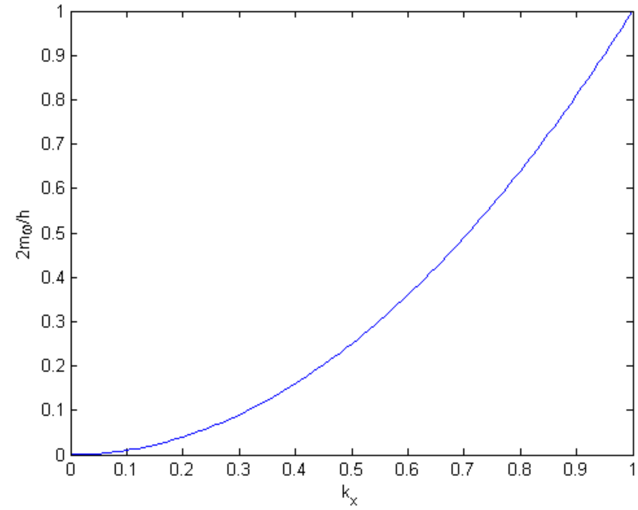

Figure 9: Dispersion diagram.

solution of the form,

$$\begin{aligned} \Psi(x,y,z,t) &= e^{i\mathbf{k}\cdot\mathbf{r}}e^{-i\omega t}, \\ &\Rightarrow \frac{\partial \Psi}{\partial t} = -i\omega\Psi. \end{aligned} \tag{40}$$

Since $i = \exp(i\pi/2)$, the term $\partial\Psi/\partial t$ should be $\pi/2$ phase shifted with respect to the Laplacian, $\nabla^2\Psi$.

b) We compute the dispersion relation using the plane wave solution of equation 40,

$$\begin{aligned} -(k_x^2 + k_y^2 + k_z^2) &= -\frac{2m\omega}{\hbar} \\ |\mathbf{k}|^2 &= \frac{2m\omega}{\hbar}. \end{aligned} \tag{41}$$

An example of a dispersion diagram is shown in Figure 9.

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