2.830J / 6.780J / ESD.63J Control of Manufacturing Processes (SMA 6303)
Spring 2008

Problem 1: example solution courtesy of M. Imani Nejad

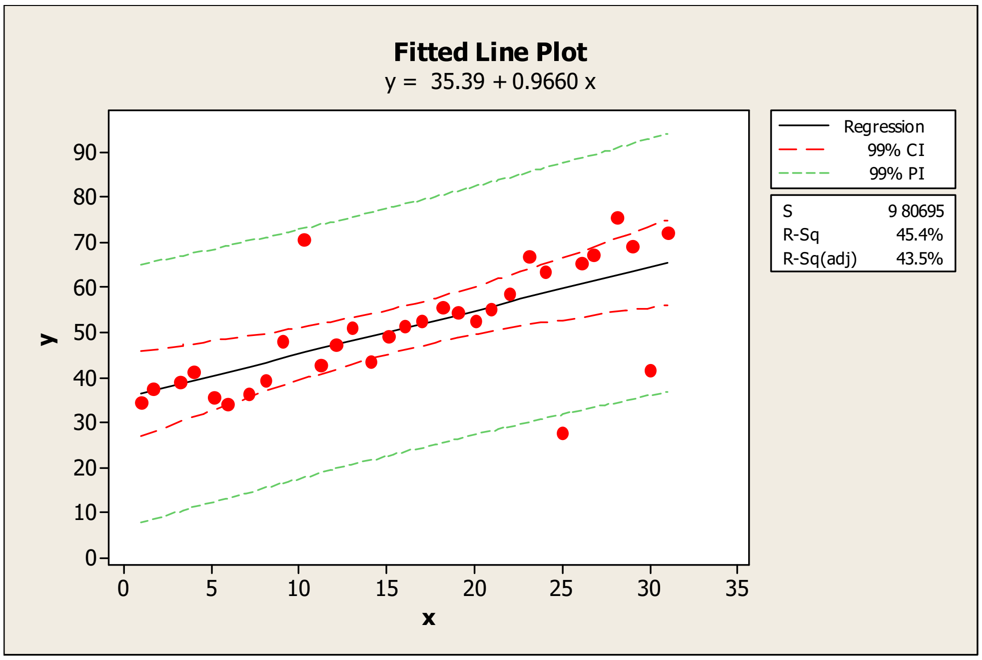
a) first we calculate the averages and then use the formula in May and Spanos to find b and b_0:

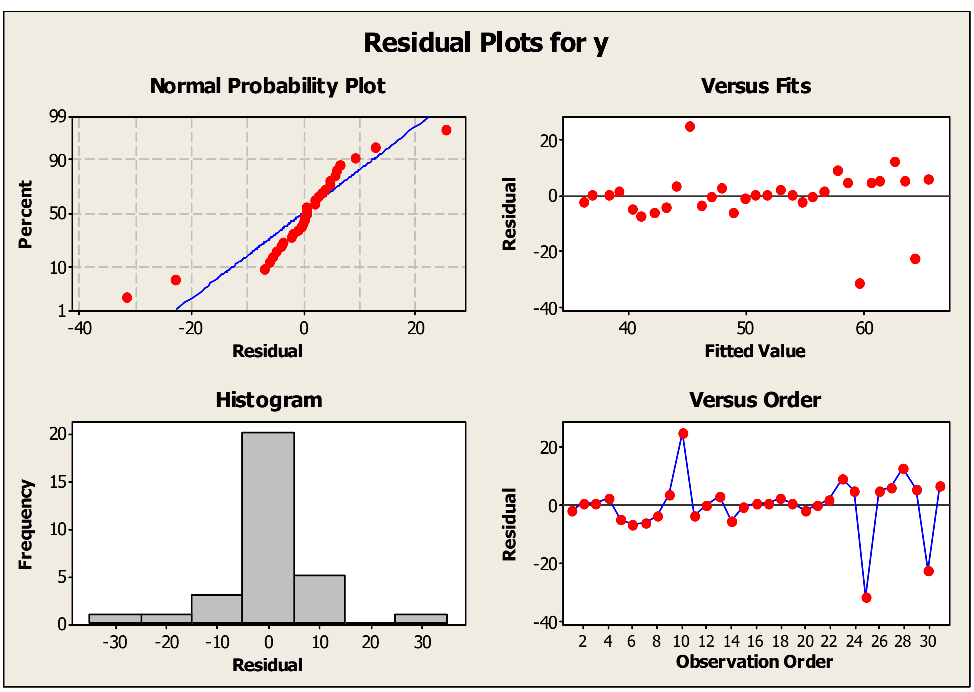
$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i = 16.02319 \qquad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = 50.86579$$

$$b = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2} = \frac{2397.779}{2482.148} = 0.96601$$

$$b_0 = \bar{y} - b\bar{x} = 50.86579 - 0.96601 * 16.02319 = 35.38724$$

So we have: $y = 35.387 + 0.966x$





In the above graph the interval is calculated based on the following formula:

$$y_0 \pm t_{\alpha/2}\sqrt{V(y)} \quad \text{where} \quad \alpha = 99\% \ \text{ and } \ V(y) = \left[\frac{1}{n} + \frac{(x-\bar{x})^2}{\sum (x-\bar{x})^2}\right] * s^2 \quad \text{For each point.}$$

```
The regression equation is
y = 35.39 + 0.9660 x

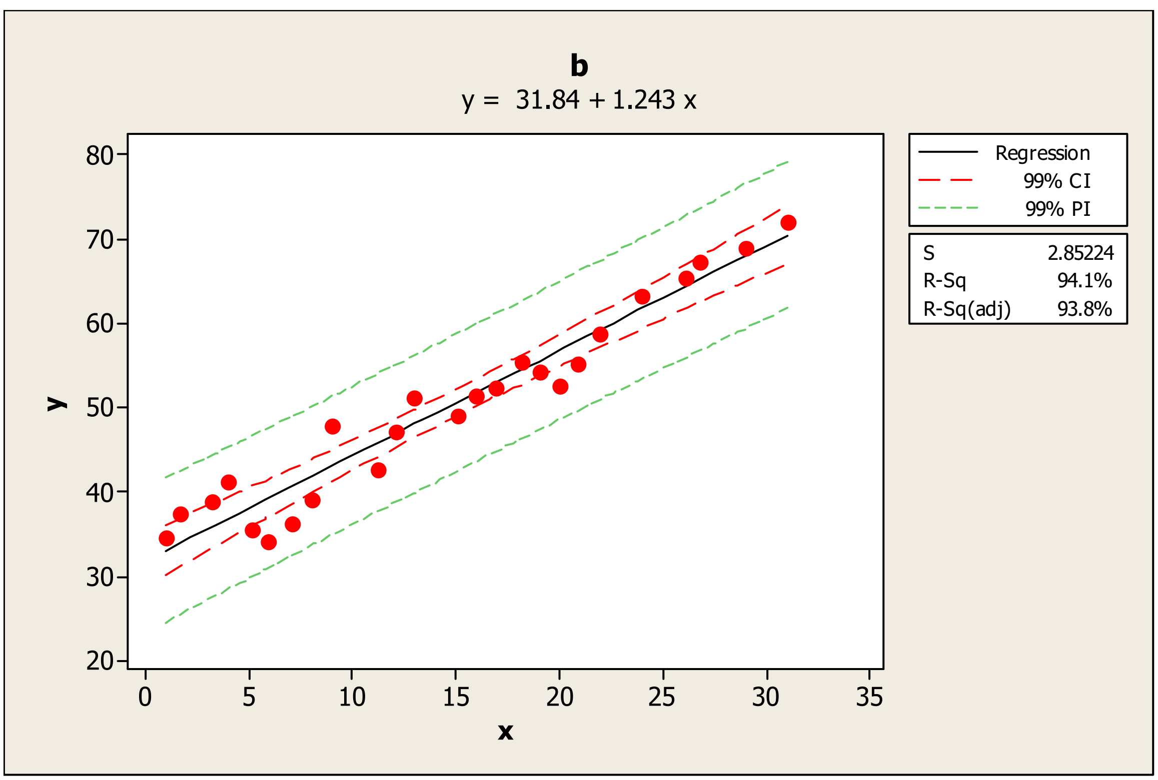

S = 9.80695   R-Sq = 45.4%   R-Sq(adj) = 43.5%

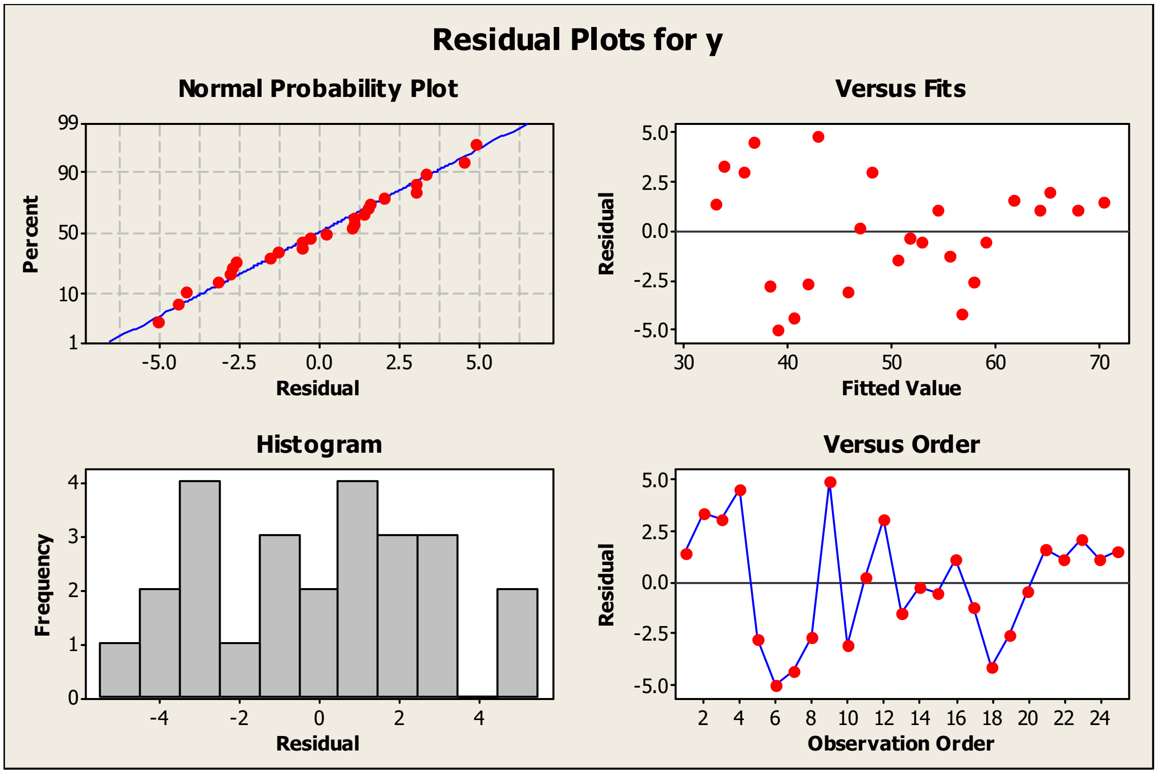

Analysis of Variance

Source      DF       SS       MS      F      P
Regression   1  2316.28  2316.28  24.08  0.000
Error       29  2789.11    96.18
Total       30  5105.39
```

b) We can see in the above graph that there are 6 points of the 99% confidence interval. So we reduce the data set and again using the formula and procedure in part (a) we get new results.

$b=1.2432 \quad b_0=31.84 \qquad y = 31.84 + 1.2432x$





Now in the reduced data, we can see that residuals are close to normal. This means that we can rely on the data more than before. Also we can make an ANOVA test which the result is as follows:

```
The regression equation is
y = 31.84 + 1.243 x


S = 2.85224   R-Sq = 94.1%   R-Sq(adj) = 93.8%


Analysis of Variance

Source      DF       SS       MS       F      P
Regression   1  2977.78  2977.78  366.03  0.000
Error       23   187.11     8.14
Total       24  3164.90
```

Also we can see that the standard error is less than before. This is a good indication that the reduced data is more reasonable. So b and b0 in reduced data is estimated better.

c) Again we reduce the data by deleting the data which are put pf 99% level of confidence. We get the following results:

```
y = 32.05 + 1.265 x
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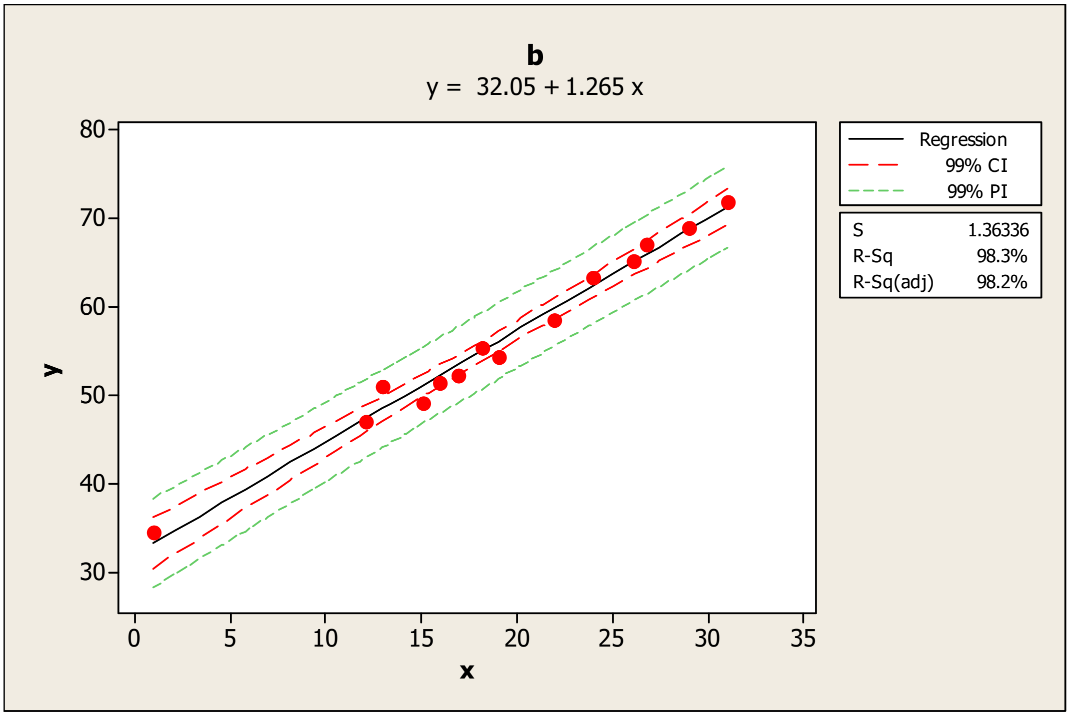

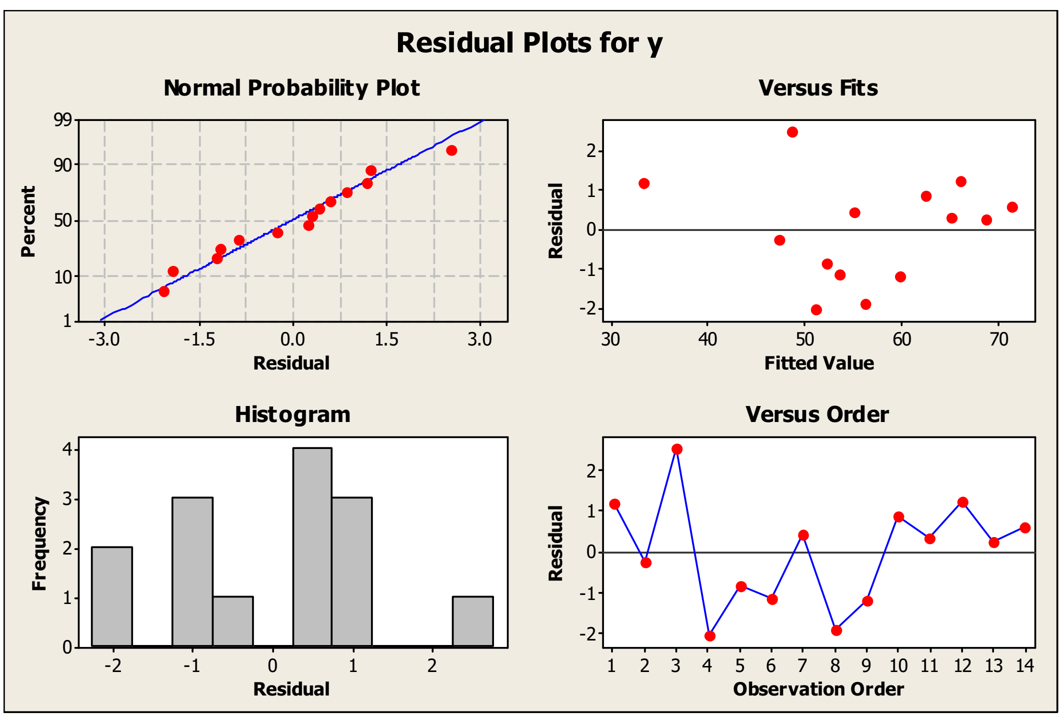

We can see that the residual are normal. But there is a little difference between new fitting model and part (b). So we perform ANOVA test and calculate the standard error.

```
The regression equation is
y = 32.05 + 1.265 x


S = 1.36336   R-Sq = 98.3%   R-Sq(adj) = 98.2%


Analysis of Variance

Source      DF       SS       MS       F      P
Regression   1  1327.56  1327.56  714.23  0.000
Error       12    22.30     1.86
Total       13  1349.87
```

So the standard error is even less than in the part (b). So if we continue this cycle we get better fitting model and finally we get a perfect fitting, but we should keep in mind that we fit the curve based on less data. This procedure can be achieved until the standard error is acceptable value.

Problem 2
This was a hard problem and effort will be generously rewarded.

First: proof of May and Spanos Eqn. 8.38:

Let the true model be:
$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$

and the estimate for **y** be:
$\hat{\mathbf{y}} = \mathbf{X}\mathbf{b}$

where **b** are estimates of model coefficients made using May and Spanos Eqn 8.37. (Note that *b* should be **b** in the text.)

Then we have:

$$\mathrm{Var}(\mathbf{b}) = \mathrm{E}[(\mathbf{b} - \boldsymbol{\beta})^{\mathsf{T}}(\mathbf{b} - \boldsymbol{\beta})] = \mathrm{E}[((\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathsf{T}}\boldsymbol{\varepsilon})^{\mathsf{T}}(\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathsf{T}}\boldsymbol{\varepsilon}]$$

$$= (\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathsf{T}}\mathrm{E}(\boldsymbol{\varepsilon}^{\mathsf{T}}\boldsymbol{\varepsilon})\mathbf{X}(\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}$$

$$= (\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathsf{T}}\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}$$

[noise is homoskedastic (has constant variance) and not autocorrelated]

$$= \sigma^2(\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathsf{T}}\mathbf{I}\mathbf{X}(\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}$$

$$= \sigma^2(\mathbf{X}^{\mathsf{T}}\mathbf{X})^{-1}$$

$$= \boldsymbol{\Sigma}_{\mathbf{b}}.$$

$\boldsymbol{\Sigma}_{\mathbf{b}}$ is the variance–covariance matrix for the estimated parameter set. Then, following the rules for expressing the variance of sums of variables, the variance of the point estimate $\hat{y}^*$ made at $\mathbf{x}^* = [1\ x_1\ x_1^2\ x_2]$ is:

$$\mathrm{Var}(\hat{y}^*) = \mathrm{Var}(\mathbf{x}^*\mathbf{b}) = \mathbf{x}^*\boldsymbol{\Sigma}_{\mathbf{b}}\mathbf{x}^{*\mathsf{T}}$$

and the confidence interval is

$$\hat{y}^* \pm t_{\alpha/2}\,[\mathrm{Var}(\hat{y}^*)]^{1/2}.$$

The number of degrees of freedom for the *t* value is [(number of data) – 4] in this case.

Thanks to Shawn Shen for his excellent work on this problem.

Problem 3 (Drain problem 4)

(a), (d)

ANOVA

Variation	d.o.f.	d.o.f.	SS	MS	F_value	Pr > F
LOT	L-1	2	0.0253	0.01267	3.18691432	0.18105
WAFER	L(W-1)	3	0.0119	0.00398	0.63273157	0.62044
SITE	LW(M-1)	6	0.0377	0.00628		
C TOTAL	LWM-1	11	0.0750	0.00681		

VARIANCE COMPONENTS

Variation Source	MS	# data in SS	# data in SS	Observed Variance	Estimated Variance	% Var
SITE	0.00628	1	1	0.00628	0.00628	79.74
WAFER	0.00398	M	2	0.00199	0.00000	0.00
LOT	0.01267	MW	4	0.00317	0.00160	20.26
TOTAL	0.00681		1	0.00681	0.00788	100.00

Interval Estimates	alpha:	0.05		
	LOWER	POINT	UPPER	d.o.f.
SITE (site to site)	0.00261	0.00628	0.03046	6
WAFER (wafer to wafer)	0.00000	0.00000	0.00000	3
LOT (lot to lot)	0.00043	0.00160	0.06306	2

(b) [courtesy M. Imani Nejad]

First line: M = 6, W = 2; second line: M = 2, W = 3:

$$S_{\bar{L}}^2 = S_L^2 + \frac{S_W^2}{W} + \frac{S_M^2}{MW} \quad \text{Then} \quad S_{\bar{L}}^2 = 0.0016 + 0 + \frac{0.006282}{6\times 2} = 0.00212$$

$$S_{\bar{L}}^2 = S_L^2 + \frac{S_W^2}{W} + \frac{S_M^2}{MW} \quad \text{Then} \quad S_{\bar{L}}^2 = 0.0016 + 0 + \frac{0.006282}{2\times 3} = 0.00264$$

(c)

The best plan would offer the highest degrees of freedom and thus the most accurate estimate. Degrees of freedom for lot variance depends on the number of wafers, so the number of wafers should be high. It is still necessary to distinguish wafer variance, though, so at least two measurements per wafer should be taken. Therefore, the cost per wafer is 2*$2+$5=$9, so three wafers in each lot should be measured.

Problem 4

a) From the lecture, for c control parameters and n noise parameters, the number of tests is given by:
Crossed array: 3^c*2^n
Response surface: $2^c+(c+1)n$

b) A half-fraction can be used when main effects are not aliased with two-factor interactions. For 2-level (linear model) experiments, this occurs for a 2^{4-1} test. The number of tests is thus:
Response surface: $2^4+(4+1)1=21$
Half-fraction response surface: $2^{4-1}+(4+1)1=13$
Crossed array: $3^4 2^1=162$

c) There must be an interaction between the noise factors and the control factors; otherwise, the process variation won't depend on the inputs.

d) Run several replicates at each of the controllable input settings. This will give two useful response variables: mean and variance. Least-square models would then be fit for the mean and the variance, and response surface methods used to find the optimum.