Inventory

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Storage

- Storage is *fundamental* in nature, management, and engineering.

  ★ In nature, *energy* is stored. Life can only exist if the acquisition of energy can occur at a different time from the the expenditure of energy.

  ★ In management, *products* are stored.

  ★ In engineering, *energy* is stored (springs, batteries, capacitors, inductors, etc.); *information* is stored (RAM and hard disks); *water* is stored (dams and reservoirs); etc.

Storage

- *The purpose of storage is to allow systems to survive even when important events are unsynchronized.* For example,
  - ⋆ the separation in time from the acquisition or production of something and its consumption; and
  - ⋆ the occurrence of an event at one location (such as a machine failure or a power surge) which can prevent desired performance or do damage at another.

- Storage improves system performance by decoupling parts of the system from one another.

Storage

- Storage decouples creation/acquisition and emission.

- It allows production systems (of energy or goods) to be built with capacity less than the peak demand.

- It reduces the propagation of disturbances, and thus reduces instability and the fragility of complex, expensive systems.

Storage

- Reduces the propagation of disturbances (eg, machine failures).

- Allows economies of scale:
  - ★ volume purchasing
  - ★ set-ups

- Helps manage seasonality and limited capacity.

- Helps manage uncertainty:
  - ★ Short term: random arrivals of customers or orders.
  - ★ Long term: Total demand for a product next year.

Storage

- Financial: raw materials paid for, but no revenue comes in until item is sold.
- Demand risk: item may not be sold due to (for example)
  - ★ time value (newspaper)
  - ★ obsolescence
  - ★ fashion
- Holding cost (warehouse space)
- Damage/theft/spoilage/loss

Newsvendor Problem

... formerly called the "Newsboy Problem".

Motive: demand risk.

Newsguy buys x newspapers at c dollars per paper (*cost*). Demand for newspapers, at *price* $r > c$ is w. Unsold newspapers are redeemed at *salvage price* $s < c$.

Demand W is a continuous random variable with distribution function $F(w) = \text{prob}(W \leq w); f(w) = dF(w)/dw$ exists for all w.

Note that $w \geq 0$ so $F(w) = 0$ and $f(w) = 0$ for $w \leq 0$.

> *Problem:* Choose x to maximize expected profit.

Newsvendor Problem

Revenue = $R = \begin{cases} rx & \text{if } x \leq w \\ rw + s(x - w) & \text{if } x > w \end{cases}$

Profit = $P = \begin{cases} (r - c)x \text{ if } x \leq w \\ rw + s(x - w) - cx \\ \quad = (r - s)w + (s - c)x \text{ if } x > w \end{cases}$

Newsvendor Problem
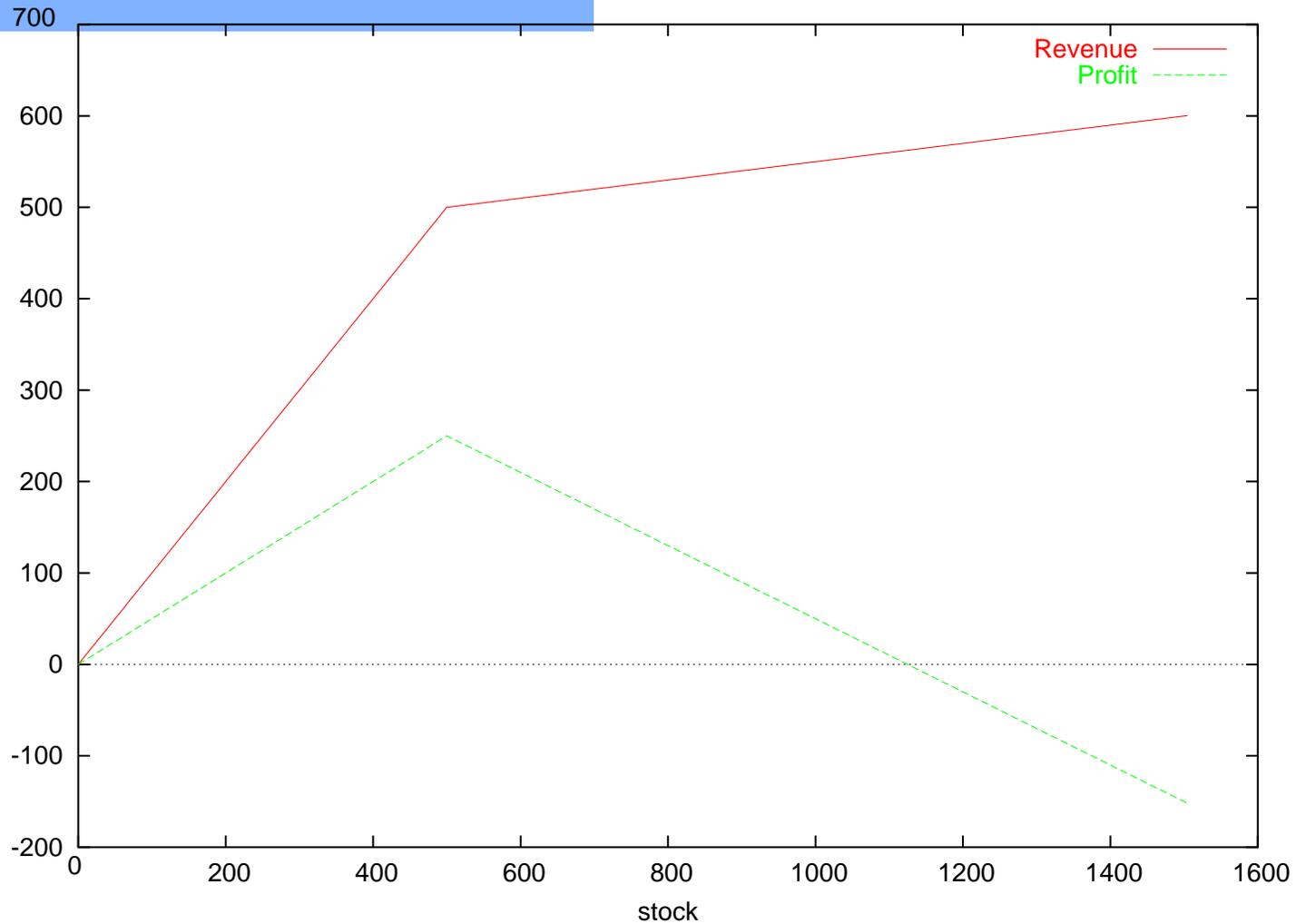
$$r = 1.; c = .5; s = .1; w = 500.$$

Newsvendor Problem

$f(w)$

$x > w$

Revenue=$rw+s(x-w)$

Profit=$rw+s(x-w)-cx$

$x < w$

Revenue=rx

Profit=$(r-c)x$

x

w
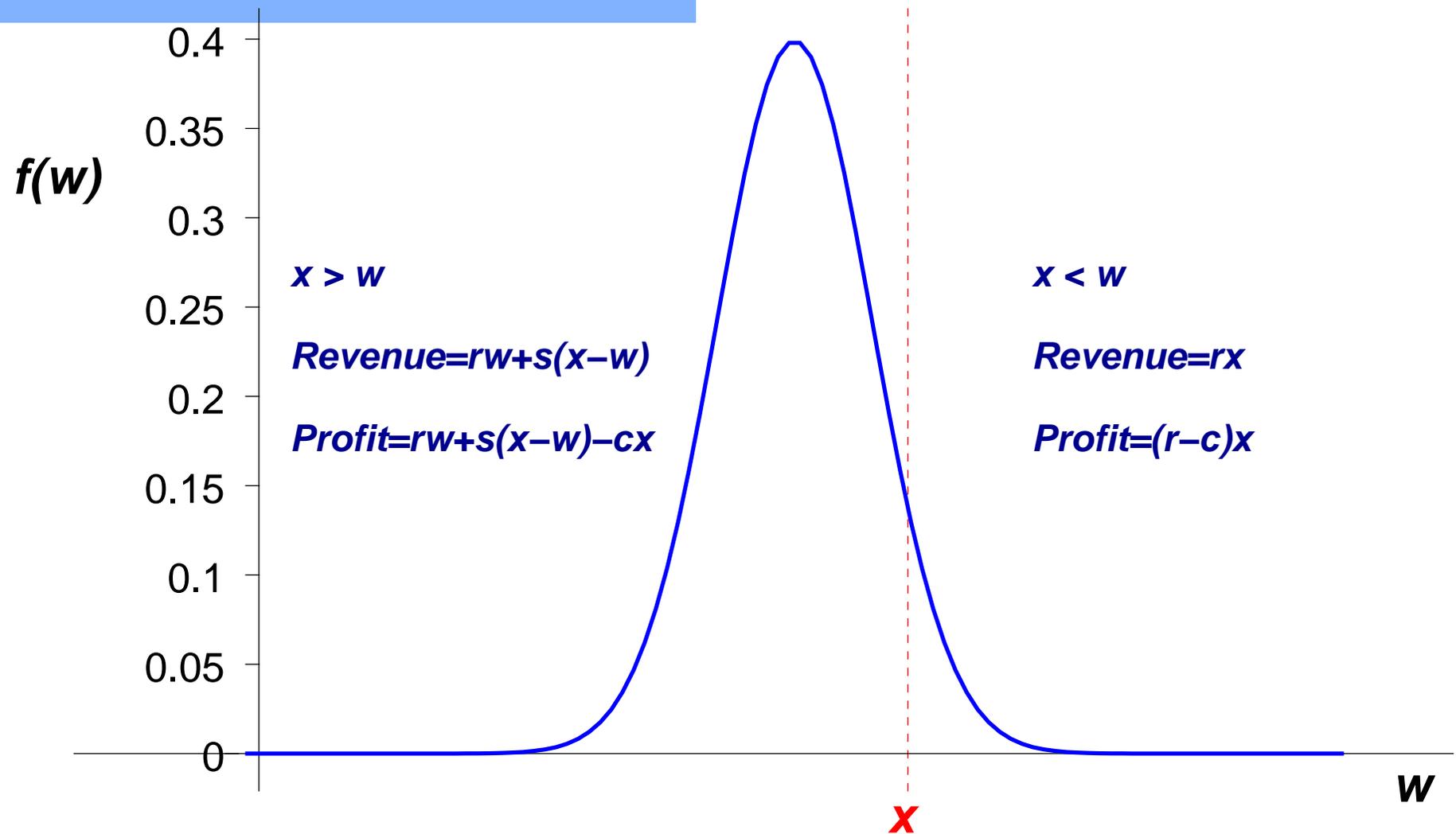
Newsvendor Problem

Expected Profit $= EP(x) =$

$$\int_{-\infty}^{x} [(r - s)w + (s - c)x] f(w) dw$$

$$+ \int_{x}^{\infty} (r - c)x f(w) dw$$

Newsvendor Problem

or $EP(x)$

$$= (r-s)\int_{-\infty}^{x} wf(w)dw + (s-c)x\int_{-\infty}^{x} f(w)dw$$

$$+(r-c)x\int_{x}^{\infty} f(w)dw$$

$$= (r-s)\int_{-\infty}^{x} wf(w)dw$$

$$+(s-c)xF(x) + (r-c)x(1-F(x))$$

Newsvendor Problem

EP is 0 when $x = 0$.

When $x \to \infty$, the first term goes to a finite constant, $(r - s)Ew$. The last term goes to 0.

The remaining term,

$$x(s - c)F(x) \to -\infty.$$

Therefore when $x \to \infty$, $EP \to -\infty$.

Newsvendor Problem

$EP(x)$

?

x
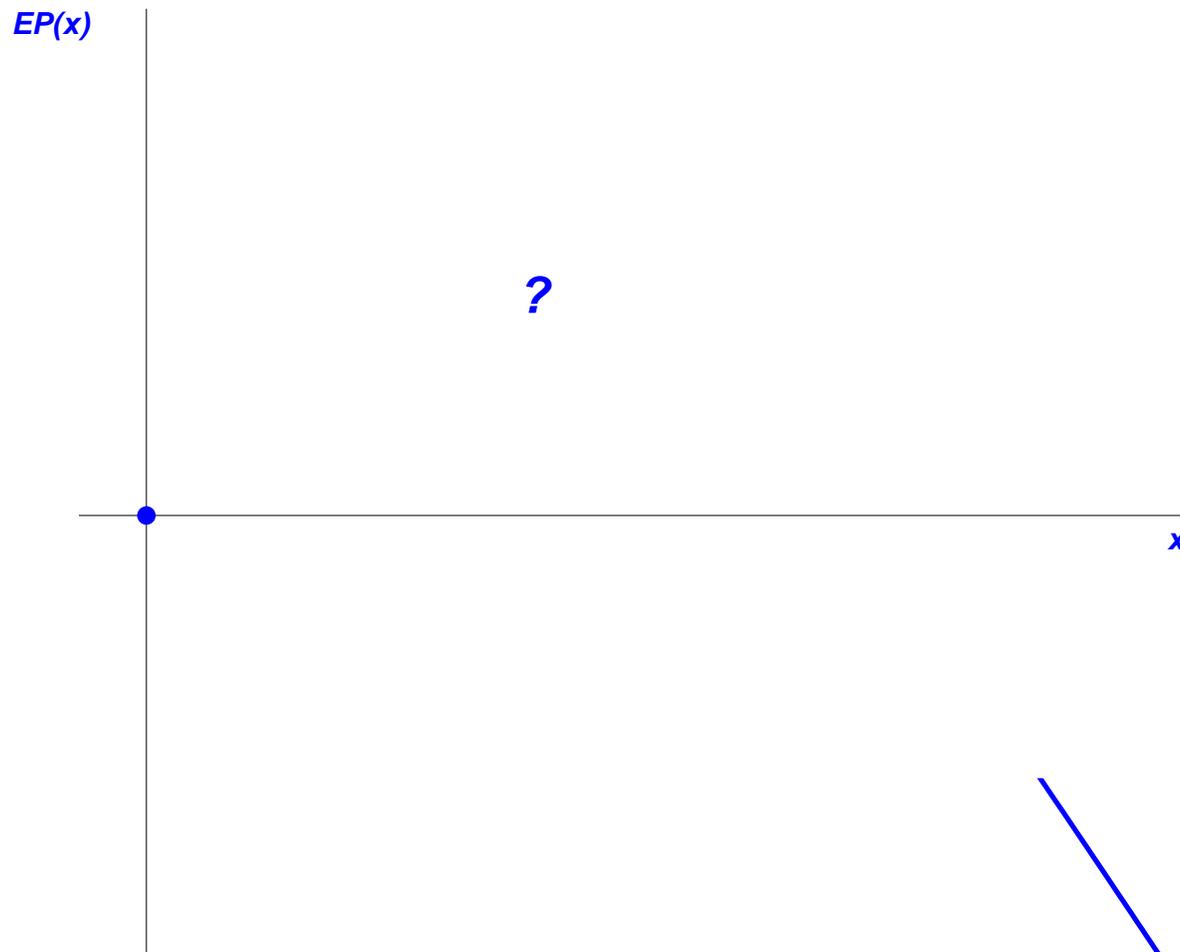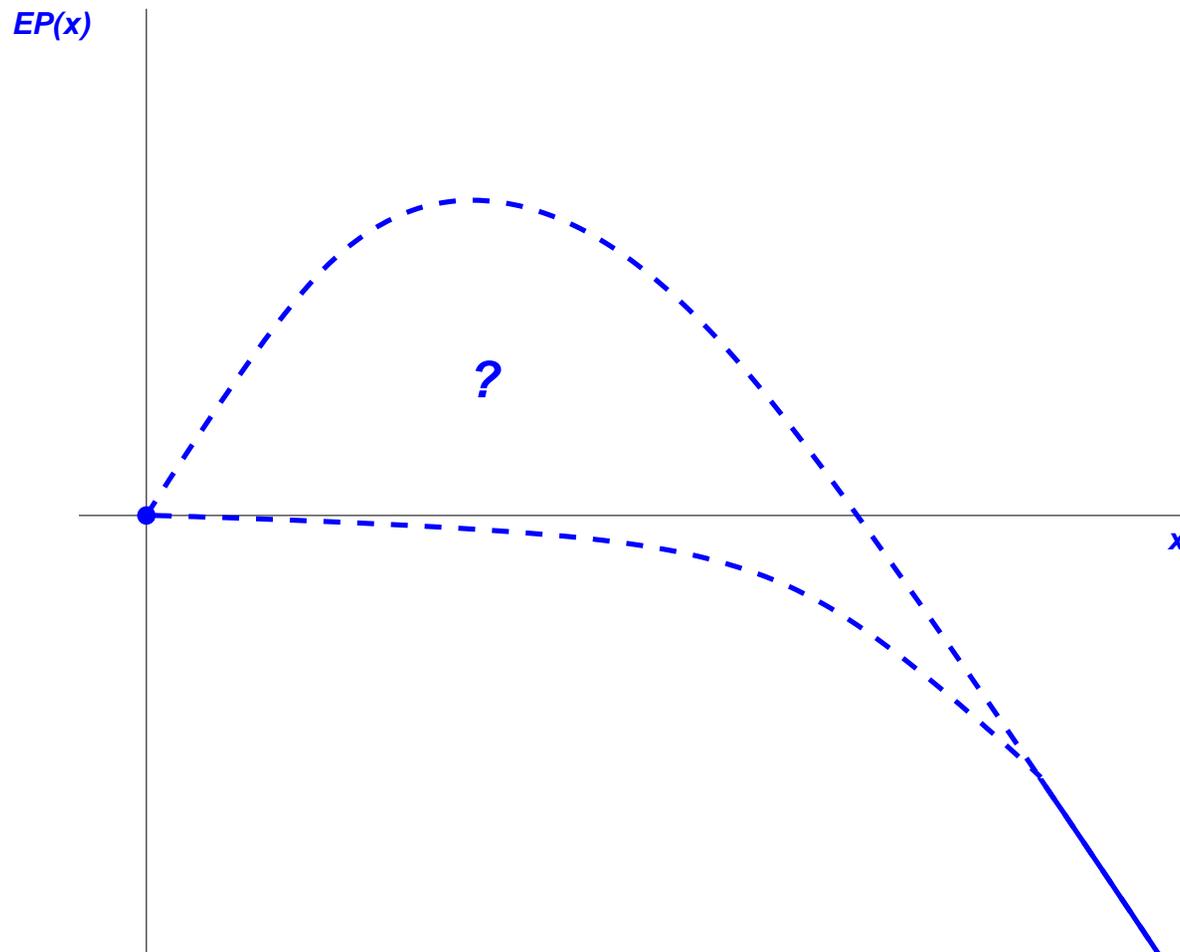
Newsvendor Problem

Newsvendor Problem

$$\frac{dEP}{dx} = (r-s)xf(x) + (s-c)F(x) + (s-c)xf(x)$$

$$+(r-c)(1-F(x)) - (r-c)xf(x)$$

$$= xf(x)(r-s+s-c-r+c)$$

$$+r-c+(s-c-r+c)F(x)$$

$$= r-c+(s-r)F(x)$$

Newsvendor Problem

Note that

- $d^2 EP/dx^2 = (s - r)f(x) \leq 0$. Therefore EP is concave and has a maximum.

- $dEP/dx > 0$ when $x = 0$. Therefore EP has a maximum which is greater than 0 for some $x = x^* > 0$.

- x^* satisfies $\dfrac{dEP}{dx}(x^*) = 0$.

Therefore $F(x^*) = \dfrac{r - c}{r - s}$.

Newsvendor Problem

Newsvendor Problem

$f(w)$

$F(x^*) = \dfrac{r-c}{r-s}$

$\mu{-}4\sigma$ $\mu{-}2\sigma$ μ x^* $\mu{+}2\sigma$ $\mu{+}4\sigma$

w
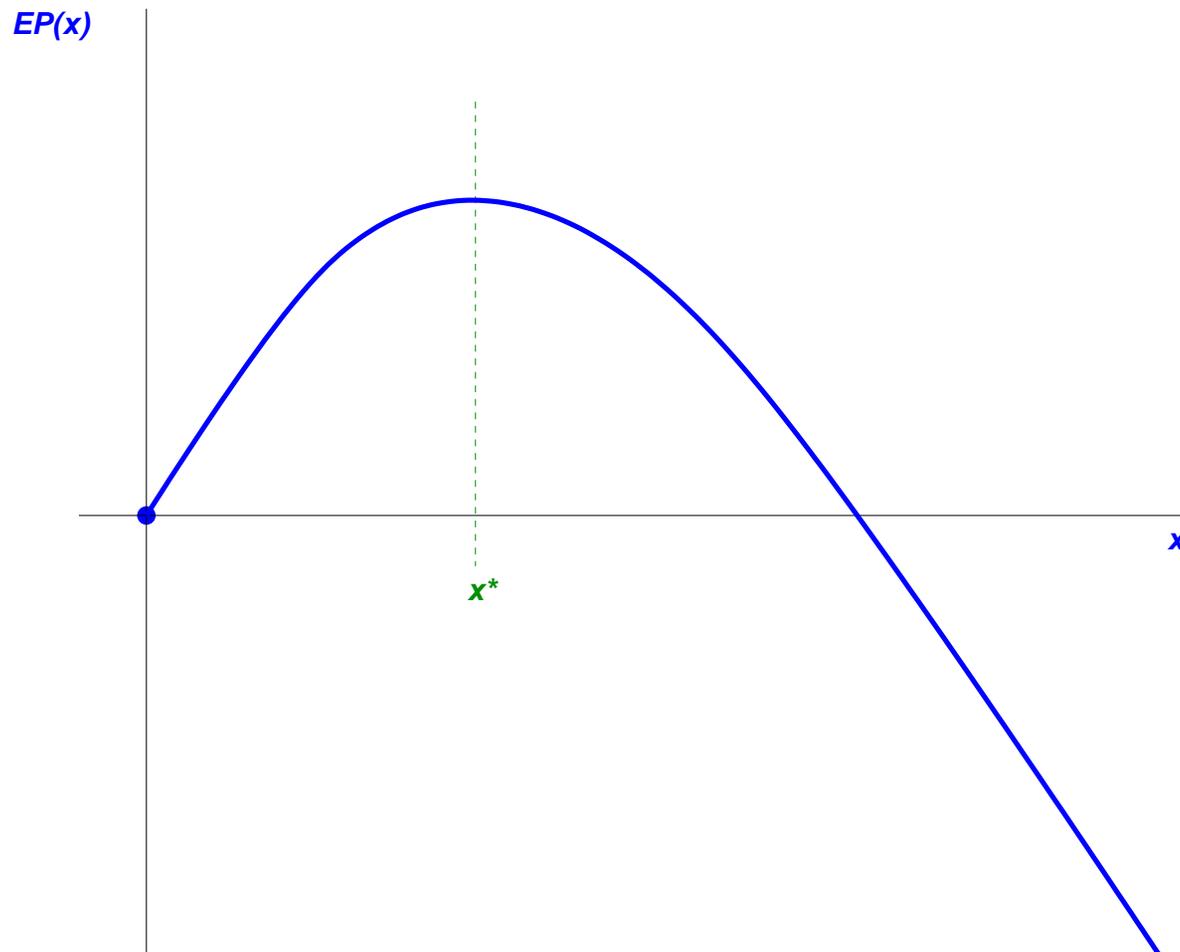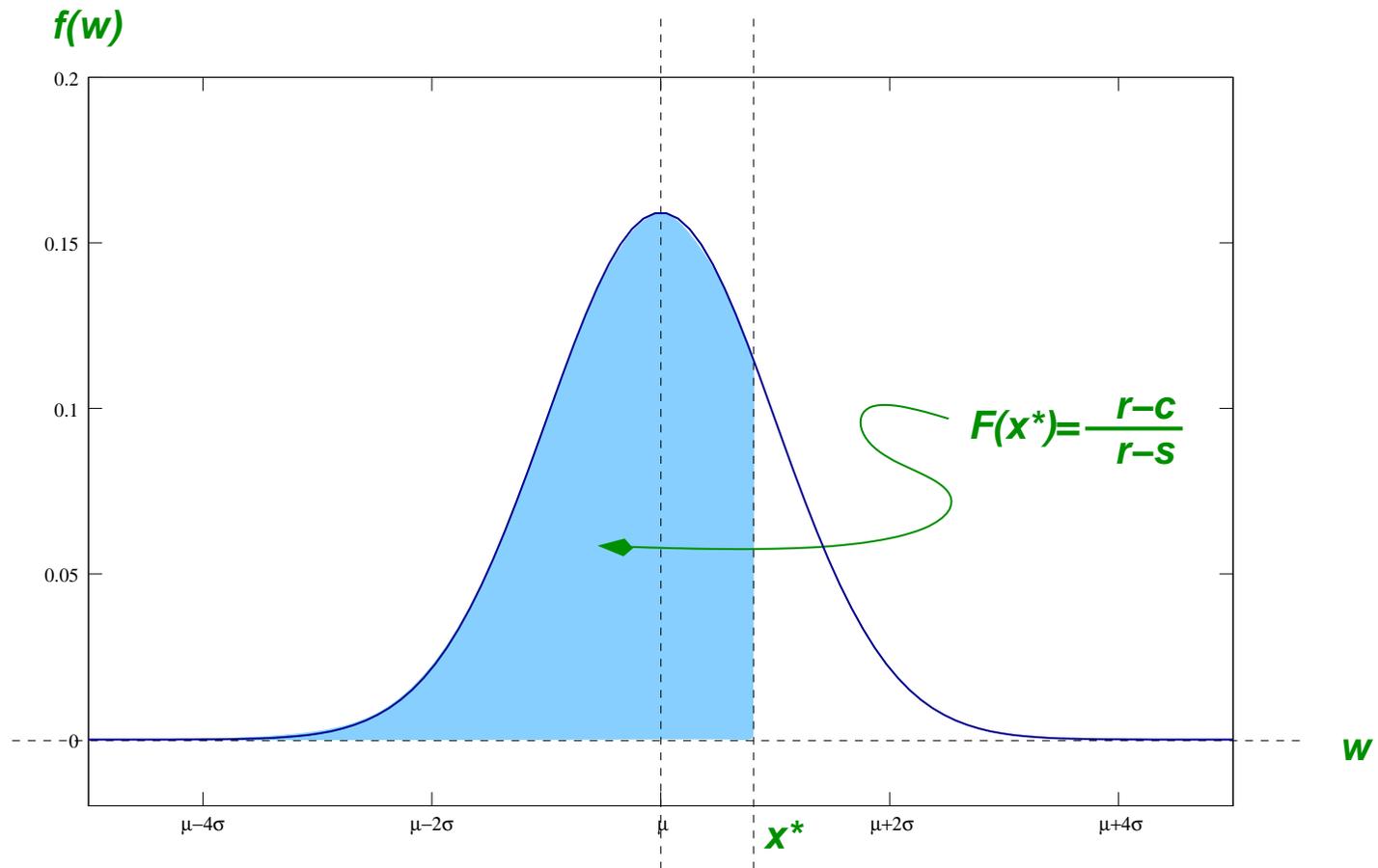
Newsvendor Problem

This can also be written

$$F(x^*) = \frac{r - c}{(r - c) + (c - s)}$$

$r - c > 0$ is the marginal profit when $x < w$.

$c - s > 0$ is the marginal loss when $x > w$.

Choose x^* so that the fraction of time you do not buy too much is

$$\frac{\text{marginal profit}}{\text{marginal profit} + \text{marginal loss}}$$

Newsvendor Problem

Example 1

$$r = 1, c = .25, s = 0, \mu_w = 100, \sigma_w = 10$$

x*   x* vs. r

Newsvendor Problem

Example 2

$$r = 1, c = .75, s = 0, \mu_w = 100, \sigma_w = 10$$

x*   x* vs. r

r

Newsvendor Problem

Example 1

$$r = 1, c = .25, s = 0, \mu_w = 100, \sigma_w = 10$$

x* VS. c

Newsvendor Problem

Example 1

$$r = 1, c = .25, s = 0, \mu_w = 100, \sigma_w = 10$$

x*   x* vs. s

Newsvendor Problem

Example 2

$$r = 1, c = .75, s = 0, \mu_w = 100, \sigma_w = 10$$

x*

x* vs. s

s

Example 1

$$r = 1, c = .25, s = 0, \mu_w = 100, \sigma_w = 10$$

x*   x* vs. Mean

Mean

Newsvendor Problem

Example 2

$$r = 1, c = .75, s = 0, \mu_w = 100, \sigma_w = 10$$

x*   x* vs. Mean

Mean

Newsvendor Problem

Example 1

$$r = 1, c = .25, s = 0, \mu_w = 100, \sigma_w = 10$$

x*   x* vs. Std

Newsvendor Problem

Example 2

$$r = 1, c = .75, s = 0, \mu_w = 100, \sigma_w = 10$$

x*   x* vs. Std

Std
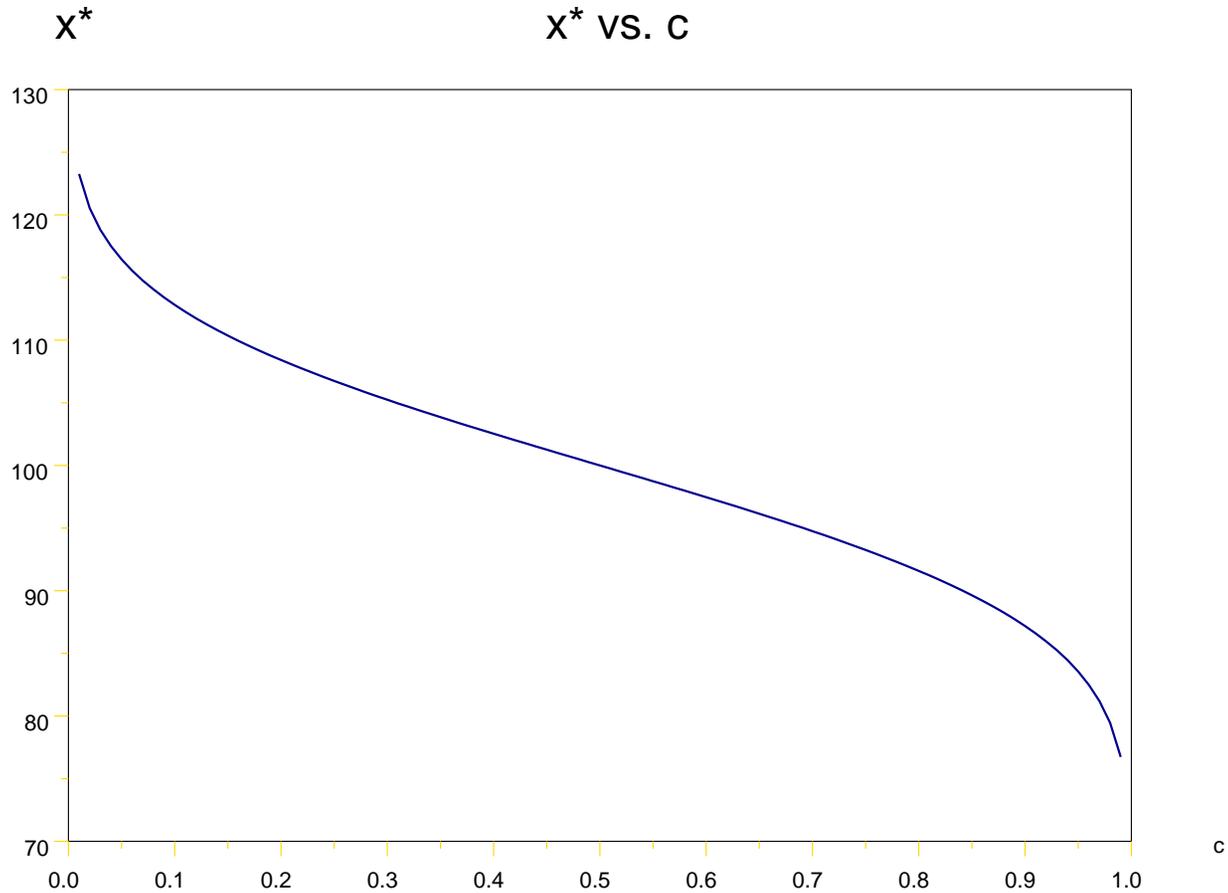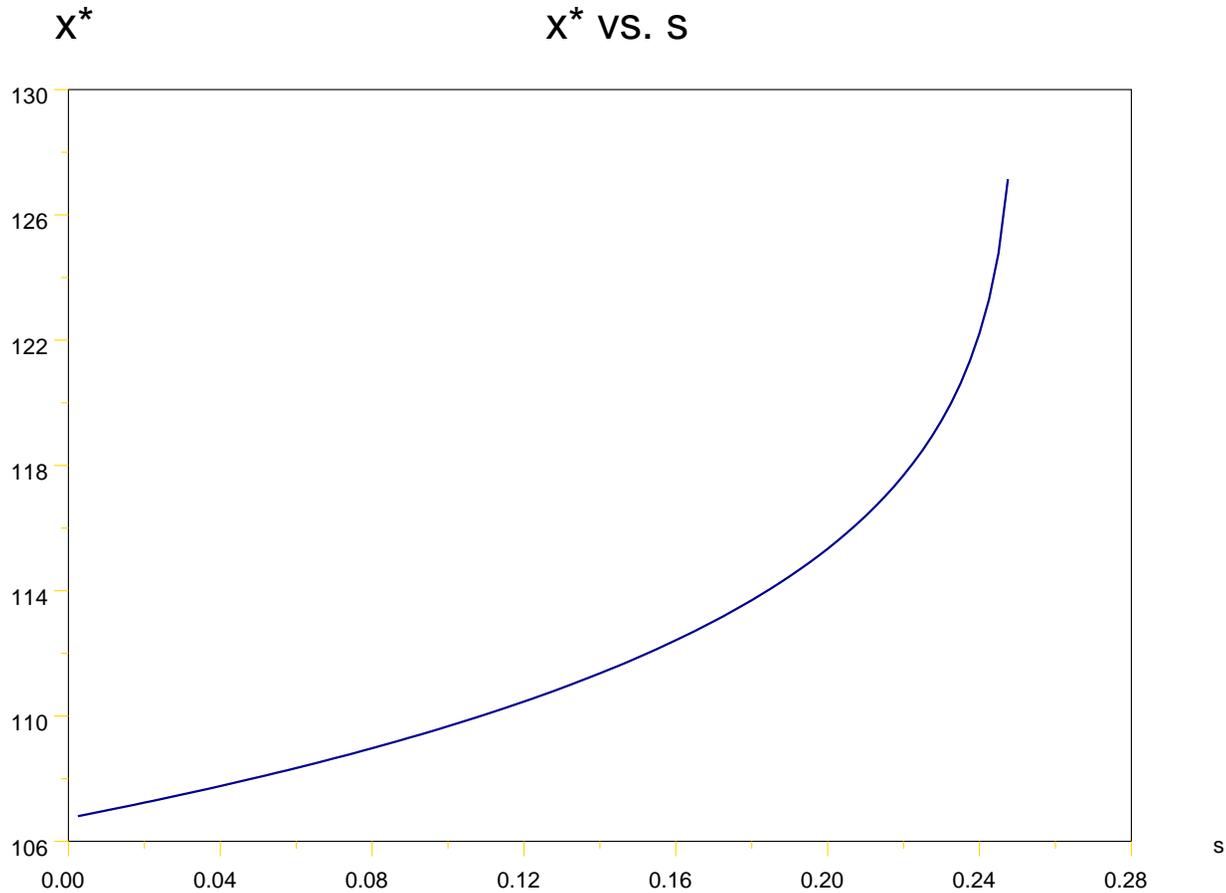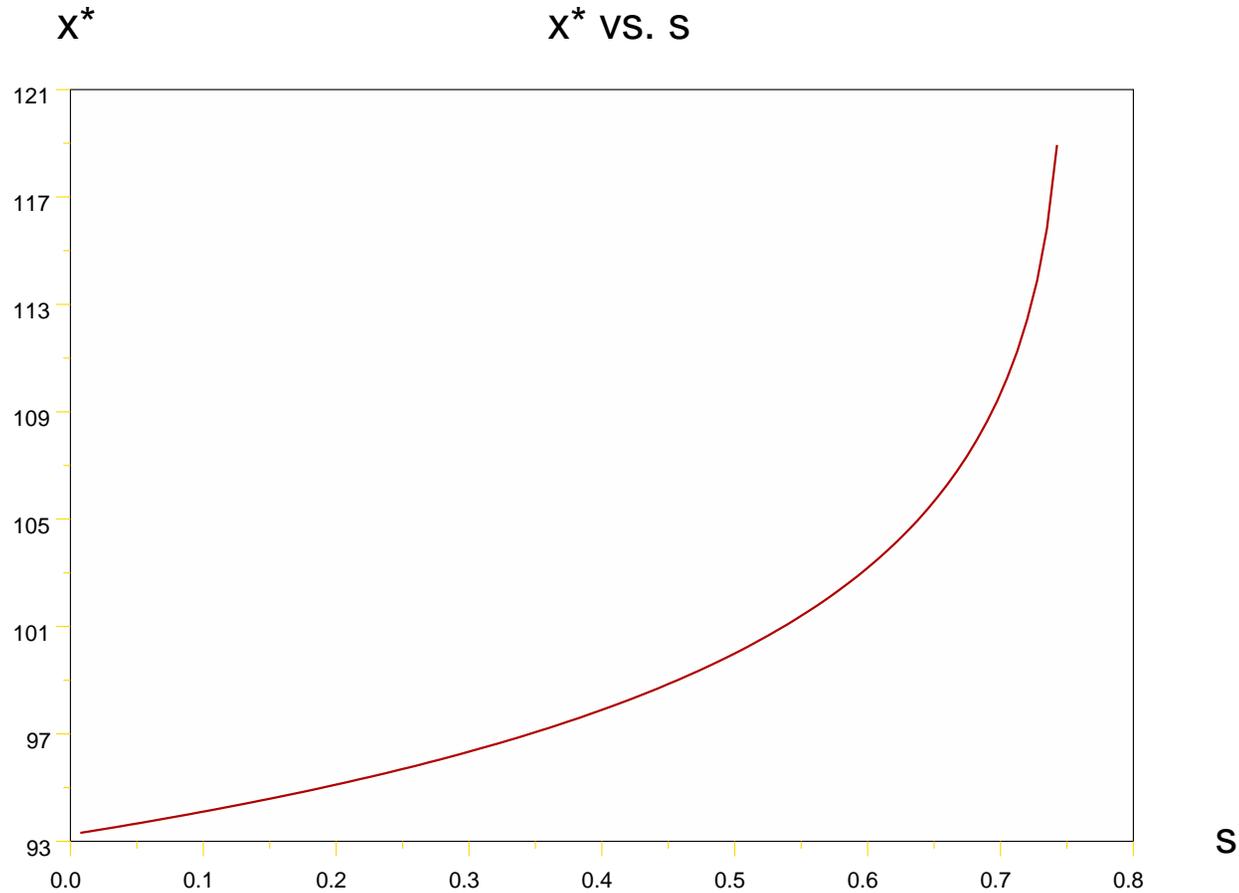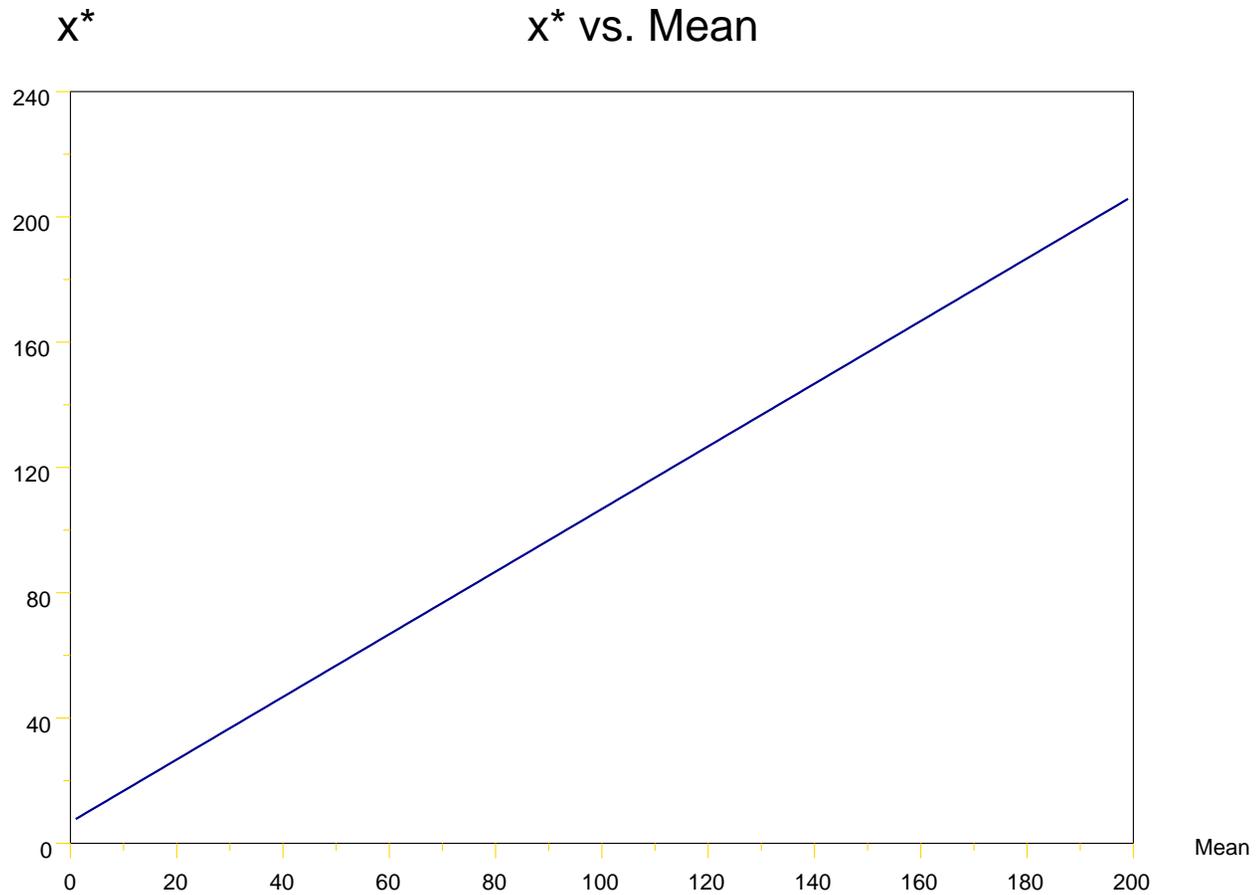
Newsvendor Problem

$$F(x^*) = \frac{r - c}{r - s}$$

Assume demand w is $N(\mu_w, \sigma_w)$. Then

$$F(w) = \Phi\left(\frac{w - \mu_w}{\sigma_w}\right)$$

where Φ is the standard normal cumulative distribution function.

Therefore

$$\Phi\left(\frac{x^* - \mu_w}{\sigma_w}\right) = \frac{r - c}{r - s}$$

or

$$\frac{x^* - \mu_w}{\sigma_w} = \Phi^{-1}\left(\frac{r - c}{r - s}\right)$$

or

$$\boxed{x^* = \mu_w + \sigma_w \Phi^{-1}\left(\frac{r - c}{r - s}\right)}$$

Why is x^* linear in μ_w and σ_w?

Note that

$$\Phi^{-1}(k) > 0 \text{ if } k > .5$$

and

$$\Phi^{-1}(k) < 0 \text{ if } k < .5$$

Therefore

- x^* increases with σ_w if $\dfrac{r - c}{r - s} > .5$, and

- x^* decreases with σ_w if $\dfrac{r - c}{r - s} < .5$.

Greater potential gain promotes riskier behavior!

EOQ

- *Economic Order Quantity*
- Motivation: economy of scale in ordering.
- Tradeoff:
  - ★ Each time an order is placed, the company incurs a fixed cost in addition to the cost of the goods.
  - ★ It costs money to store inventory.

EOQ

- No randomness.

- No delay between ordering and arrival of goods.

- No backlogs.

- Goods are required at an annual rate of λ units per year. Inventory is therefore depleted at the rate of λ units per year.

- If the company orders Q units, it must pay $s + cQ$ for the order. s is the *ordering cost* , c is the unit cost.

- It costs h to store one unit for one year. h is the *holding cost.*

EOQ

- Find a strategy for ordering materials that will minimize the total cost.

- There are two costs to consider: the ordering cost and the holding cost.

EOQ

- At time 0, inventory level is 0.

- Q units are ordered and the inventory level jumps instantaneously to Q.

- Material is depleted at rate λ.

- Since the problem is totally deterministic, we can wait until the inventory goes to zero before we order next. There is no danger that the inventory will go to zero earlier than we expect it to.

EOQ

- Because of the very simple assumptions, we can assume that the optimal strategy does not change over time.

- Therefore the policy is to order Q each time the inventory goes to zero. We must determine the optimal Q.
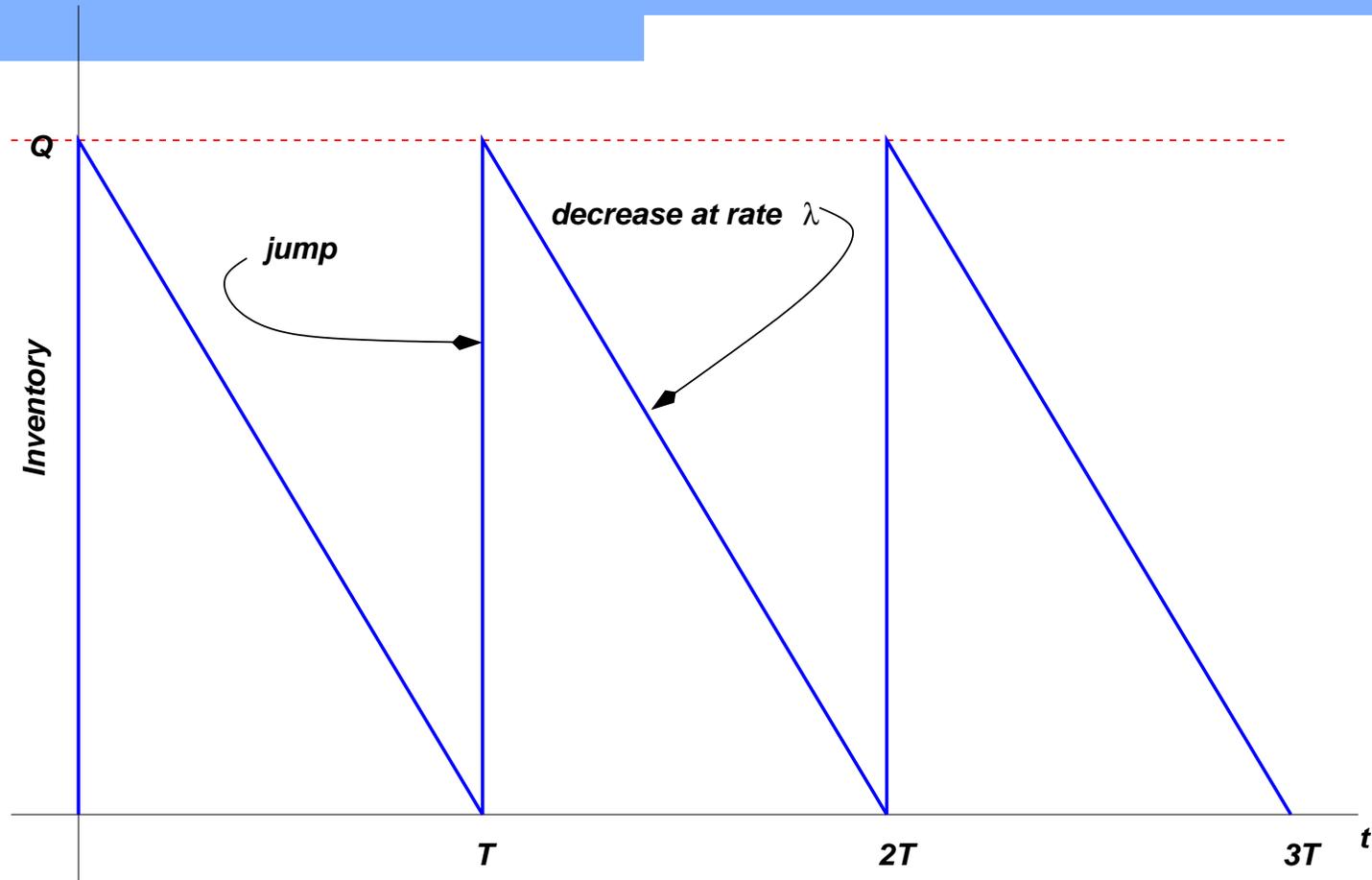
$$T = Q/\lambda \text{ (years)}$$

EOQ

- The number of orders in a year is $1/T = \lambda/Q$. Therefore, the ordering cost in a year is $s\lambda/Q$.

- The average inventory is $Q/2$. Therefore the average inventory holding cost is $hQ/2$.

- Therefore, we must minimize

$$C = \frac{hQ}{2} + \frac{s\lambda}{Q}$$

over Q.

EOQ

Then

$$\frac{dC}{dQ} = \frac{h}{2} - \frac{s\lambda}{Q^2} = 0$$

or

$$Q^* = \sqrt{\frac{2s\lambda}{h}}$$

# EOQ

In the following graphs, the base case is

- $\lambda = 3000$
- $s = .001$
- $h = 6$

Note that

$$Q^* = 1$$

EOQ

EOQ

EOQ

EOQ
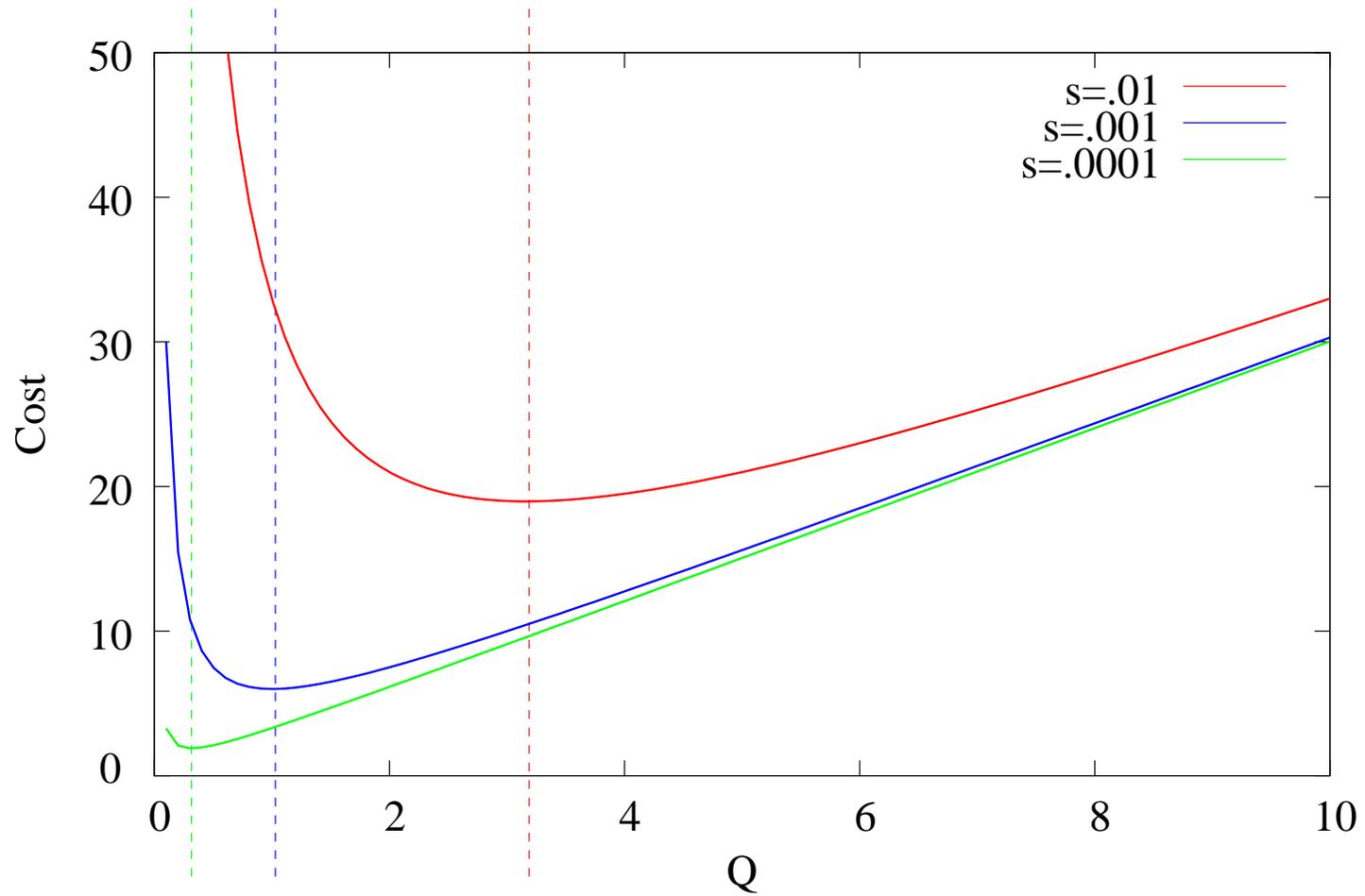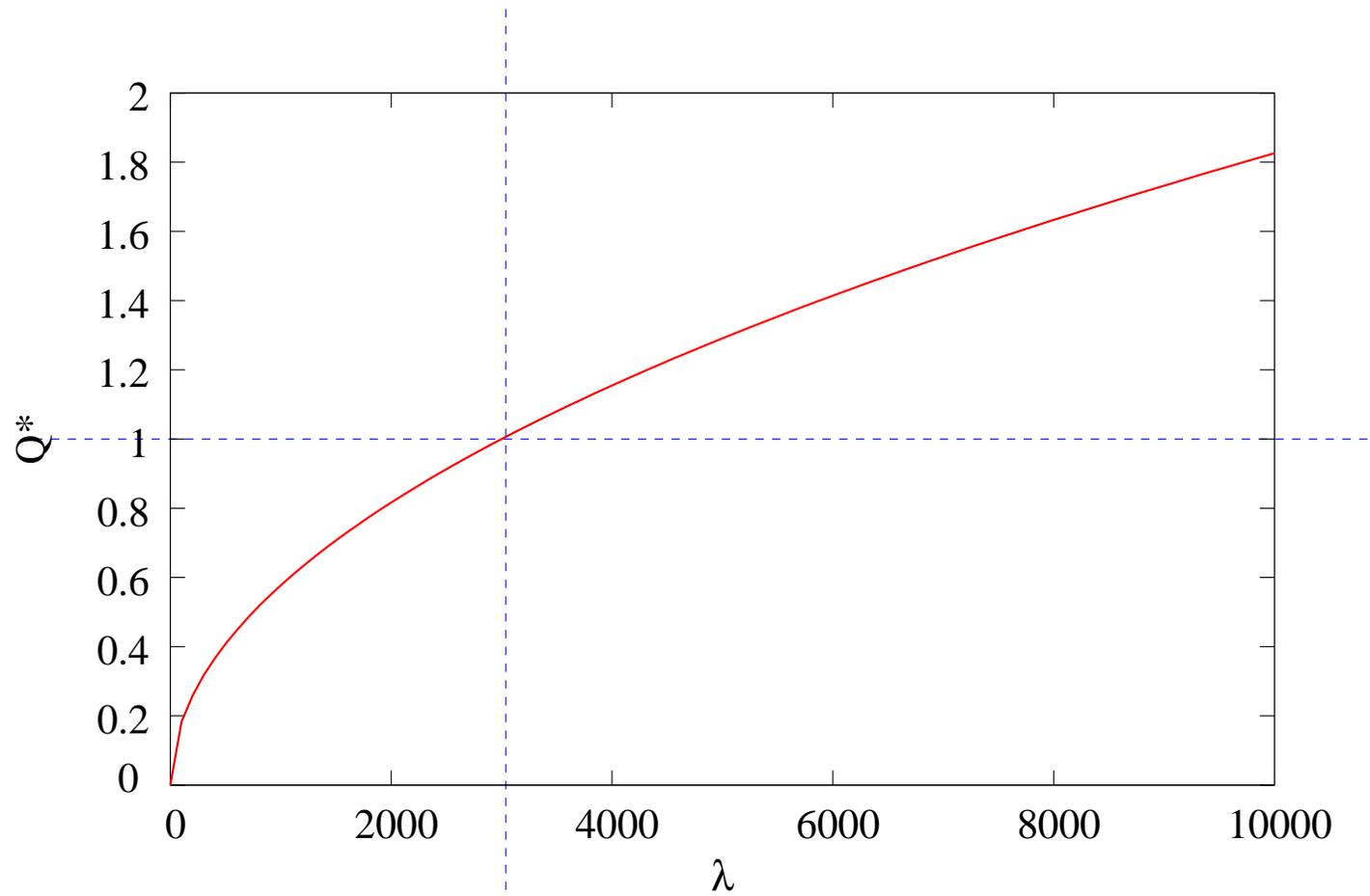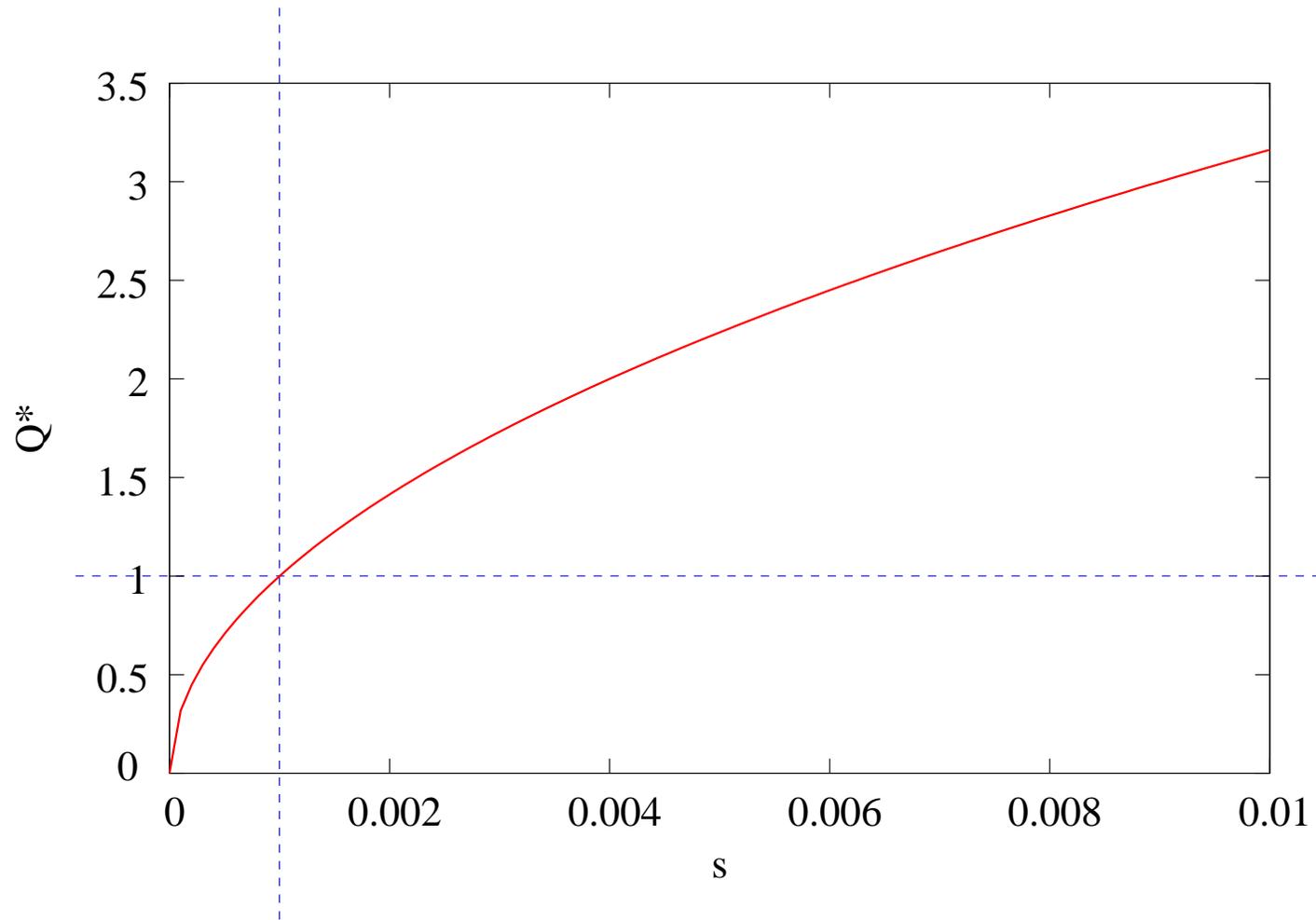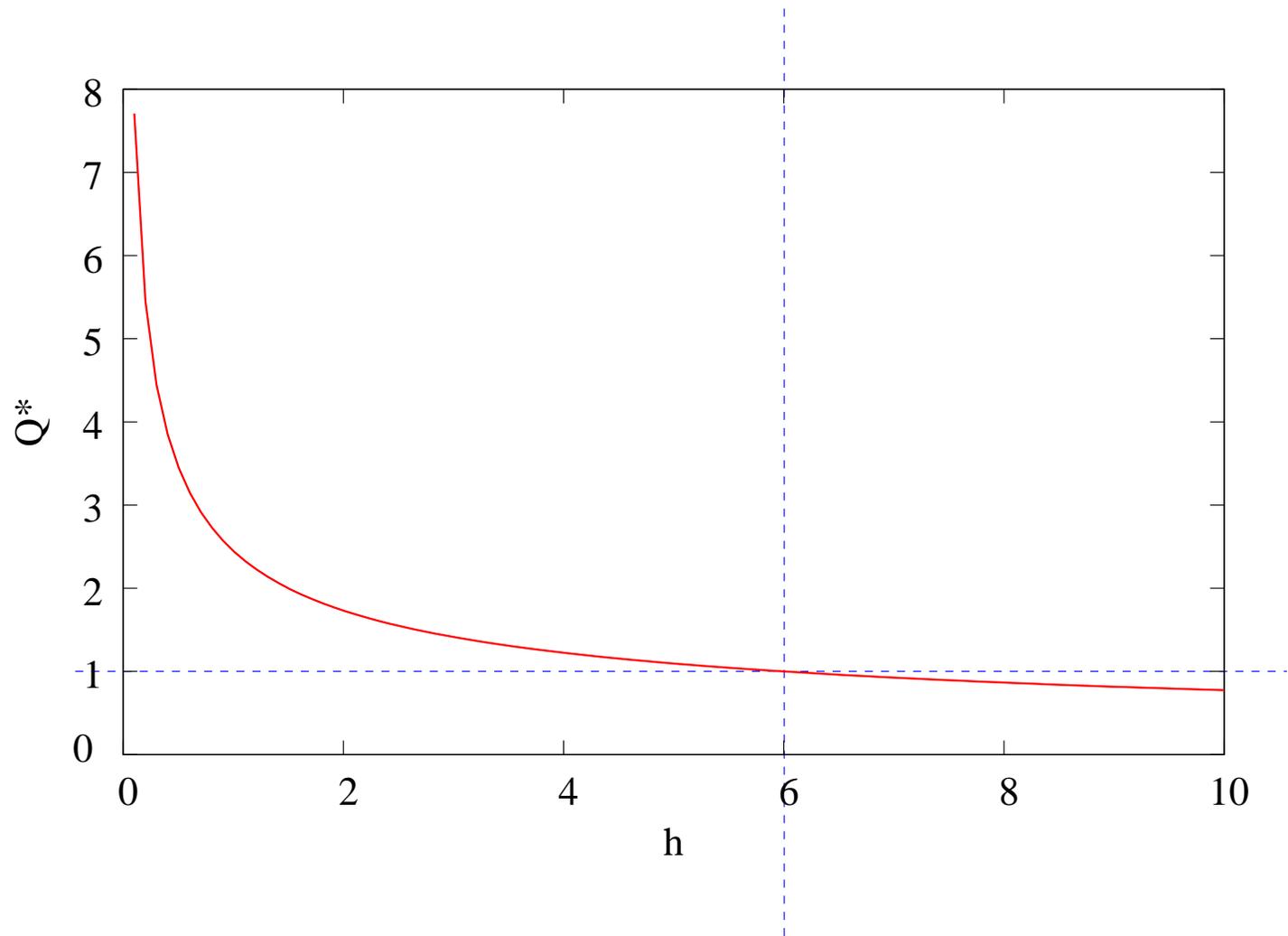
More Issues

- Random delivery times and amounts.

- Order lead time.

- Vanishing inventory.

- *Combinations of these issues and random demand, ordering/setup costs.*

Base Stock Policy

Usual assumptions:

- Demand is random.

- Inventory is held (unlike newsvendor problem).

- No ordering cost, batching, etc.

Policy

- Try to keep inventory at a fixed level S.

Issue

- Stockout: a demand occurs when there is no stock with which to satisfy it.

Base Stock Policy

Issues

- What fraction of the time will there be stockout, and how much demand occurs during stockout periods?

- How much inventory will there be, on the average?

- How much backlog will there be, on the average?

Base Stock Policy

From Supplier → **Raw Material** → **Dispatch** → **Factory Floor** → **Production** → **FG/ Backlog** → **Demand**

**Material Flow** →   **Information Flow** →
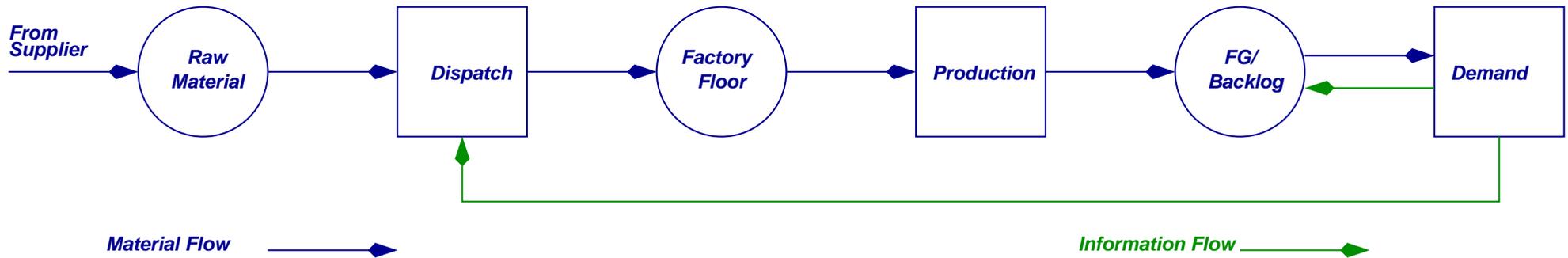
- When an order arrives (generated by "Demand"),

  ★ an item is requested from finished goods inventory (stock),

  ★ an item is delivered to the customer from the finished goods inventory (FG) *or,* if FG is empty, backlog is increased by 1, and

  ★ a signal is sent to "Dispatch" to move one item from raw material inventory to factory floor inventory.

Base Stock Policy

From Supplier → Raw Material → Dispatch → Factory Floor → Production → FG/Backlog → Demand

**Material Flow**   **Information Flow**
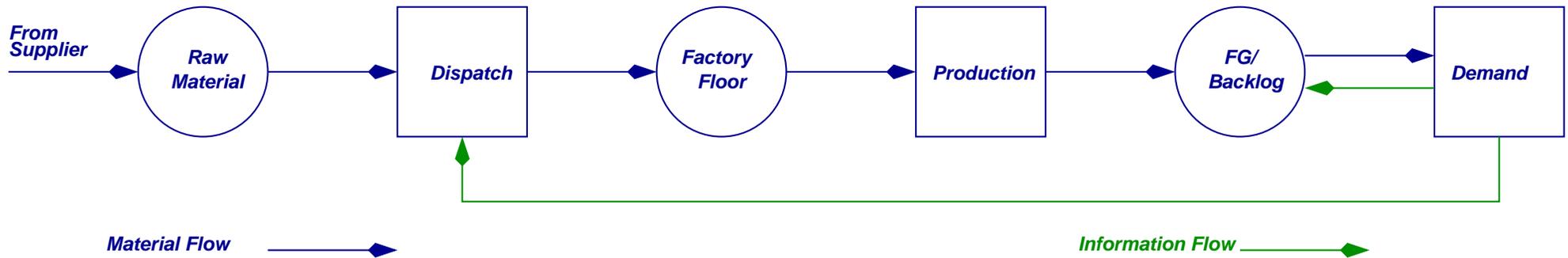
- There is some mechanism for ordering raw material from suppliers.

- We do not consider it here.

- *We assume that the raw material buffer is never empty.*

Base Stock Policy

From Supplier → Raw Material → Dispatch → Factory Floor → Production → FG/ Backlog → Demand

Material Flow

Information Flow
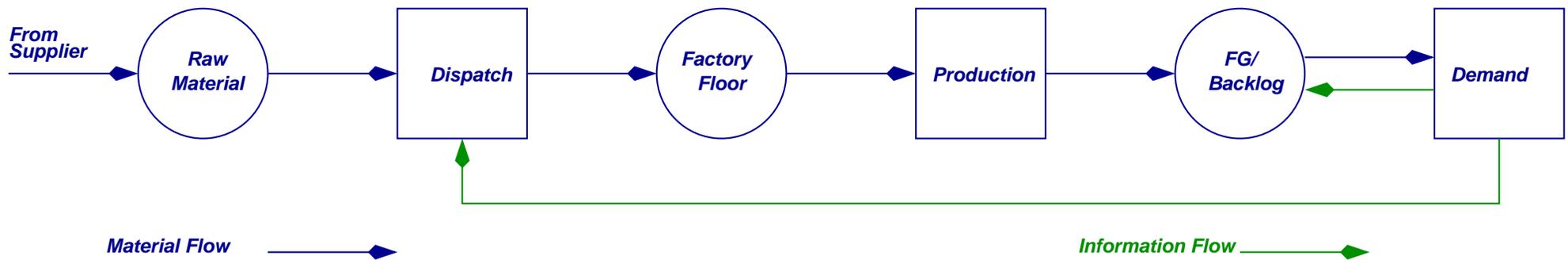
- Whenever the factory floor buffer is not empty and the production line is available, the production line takes a raw part from the factory floor buffer and starts to work on it.

- When the production line completes work on a part, it puts the finished part in the finished goods buffer *or,* if there is backlog, it sends the part to the customer and backlog is reduced by 1.
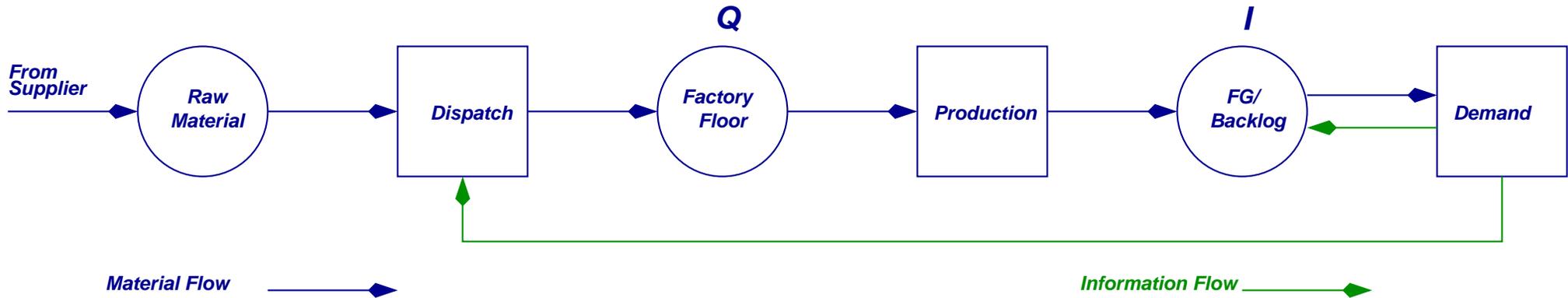
- $Q(t)$ = factory floor inventory at time t.

- $I(t) =$

  ⋆ the number of items in FG, if this is nonnegative and there is no backlog;
    or

  ⋆ −(the amount of backlog), if backlog is non-negative and FG is empty.

- There are *no lost sales.* $I(t)$ is not bounded from below. It can take on any negative value.

Base Stock Policy

Q I

From Supplier → Raw Material → Dispatch → Factory Floor → Production → FG/ Backlog → Demand

Material Flow   Information Flow
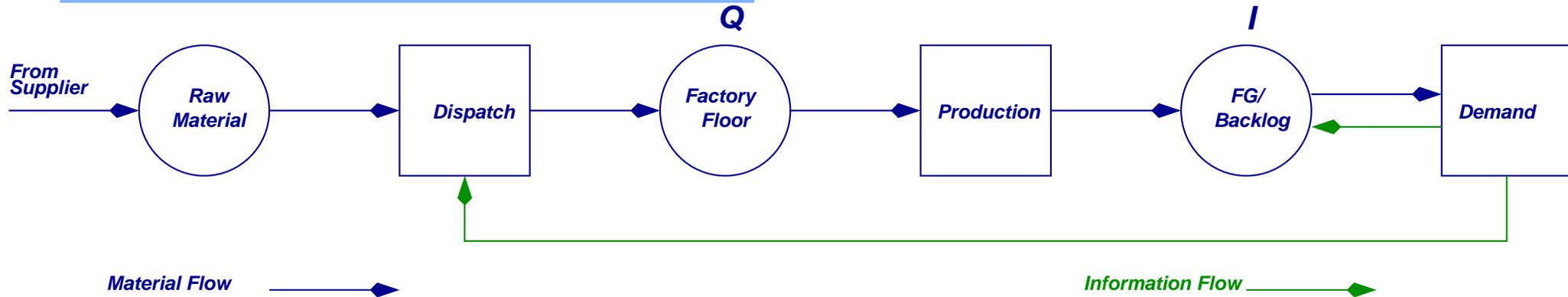
- Assume $Q(0) = 0$ and $I(0) = S$. Then $Q(0) + I(0) = S$.

- At every time t when a demand arrives,

  ⋆ $I(t)$ decreases by 1 and $Q(t)$ increases by 1 so $Q(t) + I(t)$ does not change.

- At every time t when a part is produced,

  ⋆ $I(t)$ increases by 1 and $Q(t)$ decreases by 1 so $Q(t) + I(t)$ does not change.

- Therefore $\boxed{Q(t) + I(t) = S}$ for all t.

Make-to-Stock Queue

Q

I

From Supplier → Raw Material → Dispatch → Factory Floor → Production → FG/ Backlog → Demand

Material Flow ————▶

Information Flow ————▶
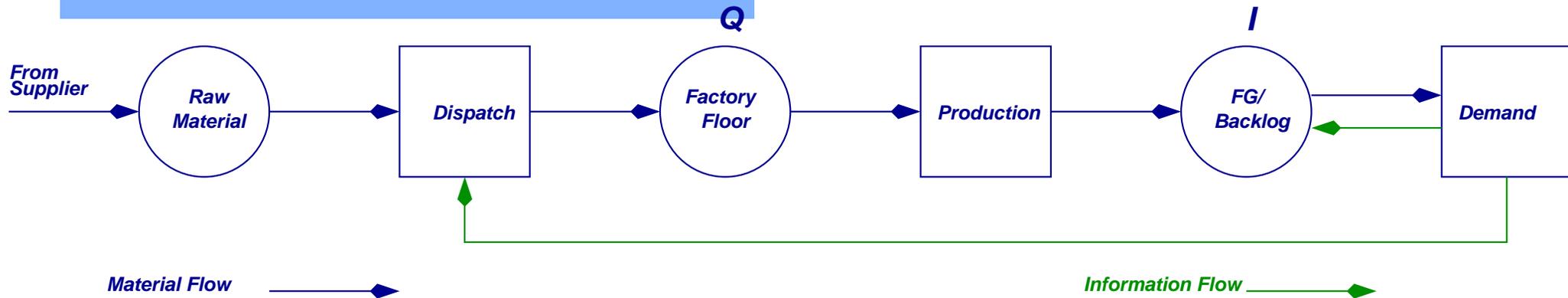
- Also, $Q(t) \geq 0$ and $I(t) \leq S$.

- Assume the factory floor buffer is infinite.

- Assume demands arrive according to a Poisson process with rate parameter λ.

- Assume the production process time is exponentially distributed with rate parameter μ.

- *Then $Q(t)$ behaves like the state of an M/M/1 queue.*

Base Stock Policy

- Assume $\lambda < \mu$.

Therefore, in steady state,

$$\text{prob}(Q = q) = (1 - \rho)\rho^q, q \geq 0$$

where $\rho = \lambda/\mu < 1$.

- *What fraction of the time will there be stockout?*
  That is, what is $\text{prob}(I \leq 0)$?

Base Stock Policy

$$\text{prob}(I \leq 0) = \text{prob}(Q \geq S) = \sum_{q=S}^{\infty} (1-\rho)\rho^q$$

$$= (1-\rho)\sum_{q=S}^{\infty} \rho^q = (1-\rho)\rho^S \sum_{q=0}^{\infty} \rho^q = (1-\rho)\rho^S \frac{1}{1-\rho}$$

so

$$\text{prob}(I \leq 0) = \rho^S$$

- *How much demand occurs during stockout periods?*

$$\rho^S \lambda$$

Base Stock Policy

- *How much inventory will there be, on the average?* (It depends on what you mean by "inventory".)

- $$EQ = \frac{\rho}{1 - \rho}$$

- $EI = S - EQ$ is *not* the expected finished goods inventory. It is the expected inventory/backlog.

- We want to know

$$E(I^+) = E \begin{cases} I & \text{if } I > 0 \\ \\ 0 & \text{otherwise} \end{cases} = E(I|I > 0)\text{prob}(I > 0)$$

Base Stock Policy

$$E(I|I > 0) = \sum_{i=1}^{S} i \, \mathrm{prob}(I = i | I > 0)$$

$$= \sum_{i=1}^{S} \frac{i \, \mathrm{prob}(I = i \cap I > 0)}{\mathrm{prob}(I > 0)} = \frac{\sum_{i=1}^{S} i \, \mathrm{prob}(I = i)}{\sum_{j=1}^{S} \mathrm{prob}(I = j)}$$

$$= \frac{\sum_{q=0}^{S-1} (S - q) \, \mathrm{prob}(Q = q)}{\sum_{s=0}^{S-1} \mathrm{prob}(Q = s)}$$

where, because $Q = S - I$, we substitute $q = S - i$ and $s = S - j$.

$$= \frac{\displaystyle\sum_{q=0}^{S-1} S \ \text{prob}(Q=q) - \sum_{q=0}^{S-1} q \ \text{prob}(Q=q)}{\displaystyle\sum_{q'=0}^{S-1} \text{prob}(Q=q')}$$

$$= S - \frac{\displaystyle\sum_{q=0}^{S-1} q \ \text{prob}(Q=q)}{\displaystyle\sum_{q'=0}^{S-1} \text{prob}(Q=q')} = S - \frac{(1-\rho)\displaystyle\sum_{q=0}^{S-1} q\rho^q}{(1-\rho)\displaystyle\sum_{q=0}^{S-1} \rho^q} = S - \frac{\displaystyle\sum_{q=0}^{S-1} q\rho^q}{\displaystyle\sum_{q=0}^{S-1} \rho^q}$$

Base Stock Policy

Therefore

$$E(I|I > 0) = S - \frac{\sum_{q=0}^{S-1} q\rho^q}{\sum_{q=0}^{S-1} \rho^q}$$

and

$$E(I^+) = S(1 - \rho)\sum_{q=0}^{S-1} \rho^q - (1 - \rho)\sum_{q=0}^{S-1} q\rho^q$$

$$= S\,\text{prob}(Q < S) - (1 - \rho)\sum_{q=0}^{S-1} q\rho^q$$

Base Stock Policy

- *How much backlog will there be, on the average?*

- This time, we are looking for $-E(I^-)$, where

$$E(I^-) = E \begin{cases} I & \text{if } I \leq 0 \\ \\ 0 & \text{otherwise} \end{cases} = E(I|I \leq 0)\text{prob}(I \leq 0)$$

$$E(I|I \leq 0) = \sum_{i=0}^{-\infty} i\,\text{prob}(I = i|I \leq 0)$$

Base Stock Policy

$$= \sum_{i=0}^{-\infty} i \, \frac{\text{prob}(I = i \cap I < 0)}{\text{prob}(I < 0)} = \frac{\sum_{i=0}^{-\infty} i \, \text{prob}(I = i)}{\sum_{j=0}^{-\infty} \text{prob}(I = j)}$$

$$= \frac{\sum_{q=S}^{\infty} (S - q) \, \text{prob}(Q = q)}{\sum_{s=S}^{\infty} \text{prob}(Q = s)}$$

where, because $Q = S - I$, we substitute $q = S - i$ and $s = S - j$.

$$= \frac{\sum_{q=S}^{\infty} S \operatorname{prob}(Q = q) - \sum_{q=S}^{\infty} q \operatorname{prob}(Q = q)}{\sum_{q'=S}^{\infty} \operatorname{prob}(Q = q')}$$

$$= S - \frac{\sum_{q=S}^{\infty} q \operatorname{prob}(Q = q)}{\sum_{q'=S}^{\infty} \operatorname{prob}(Q = q')} = S - \frac{EQ - \sum_{q=0}^{S-1} q \operatorname{prob}(Q = q)}{1 - \sum_{q'=0}^{S-1} \operatorname{prob}(Q = q')}$$

Base Stock Policy

$$= S - \frac{EQ - (1 - \rho)\sum_{q=0}^{S-1} q\rho^q}{1 - (1 - \rho)\sum_{q=0}^{S-1} \rho^q}$$

SO

$$E(I^-) = E(I|I \leq 0)\text{prob}(I \leq 0)$$

Note: $E(I^+) + E(I^-) + EQ = S.$

Base Stock Policy
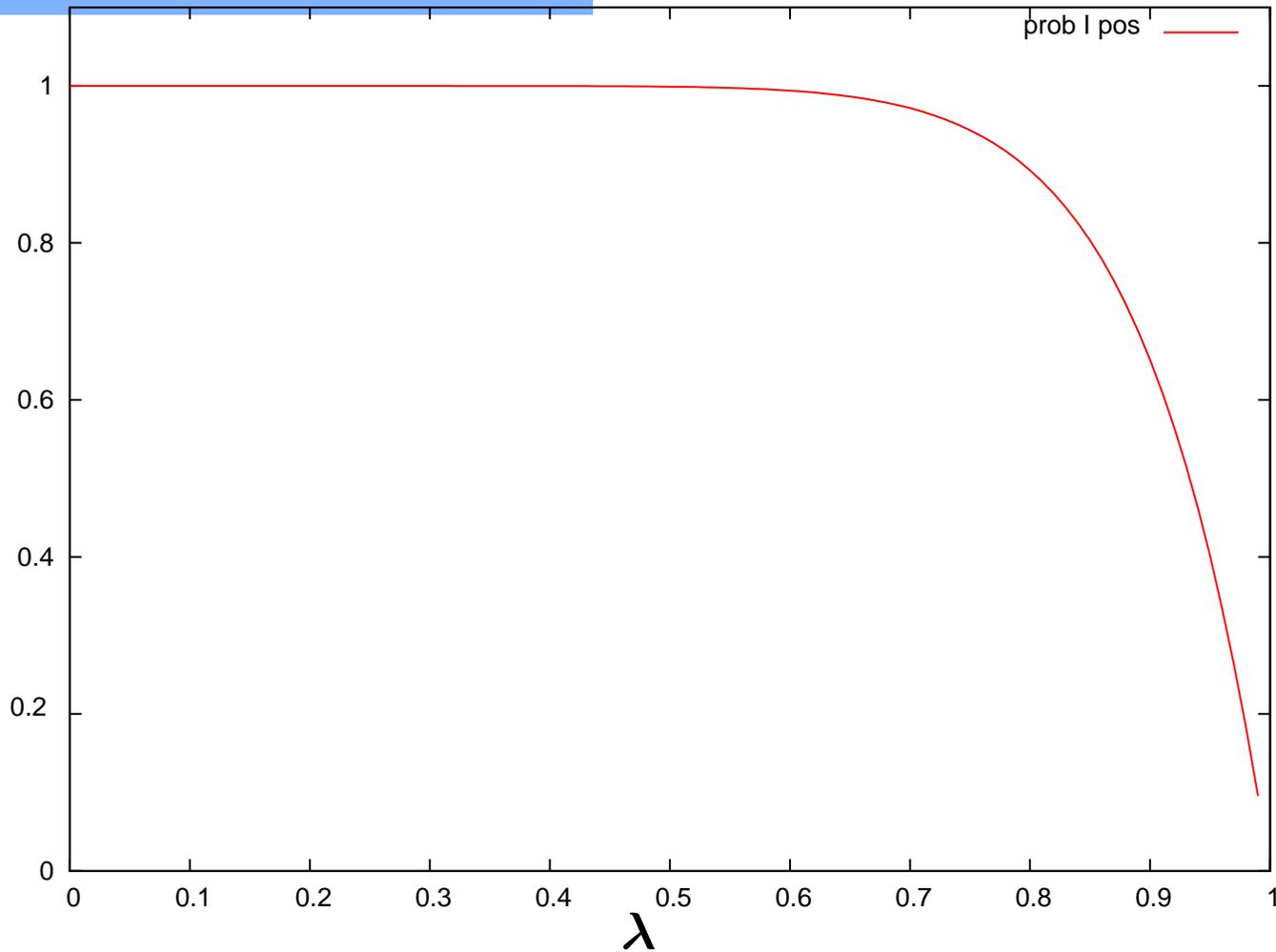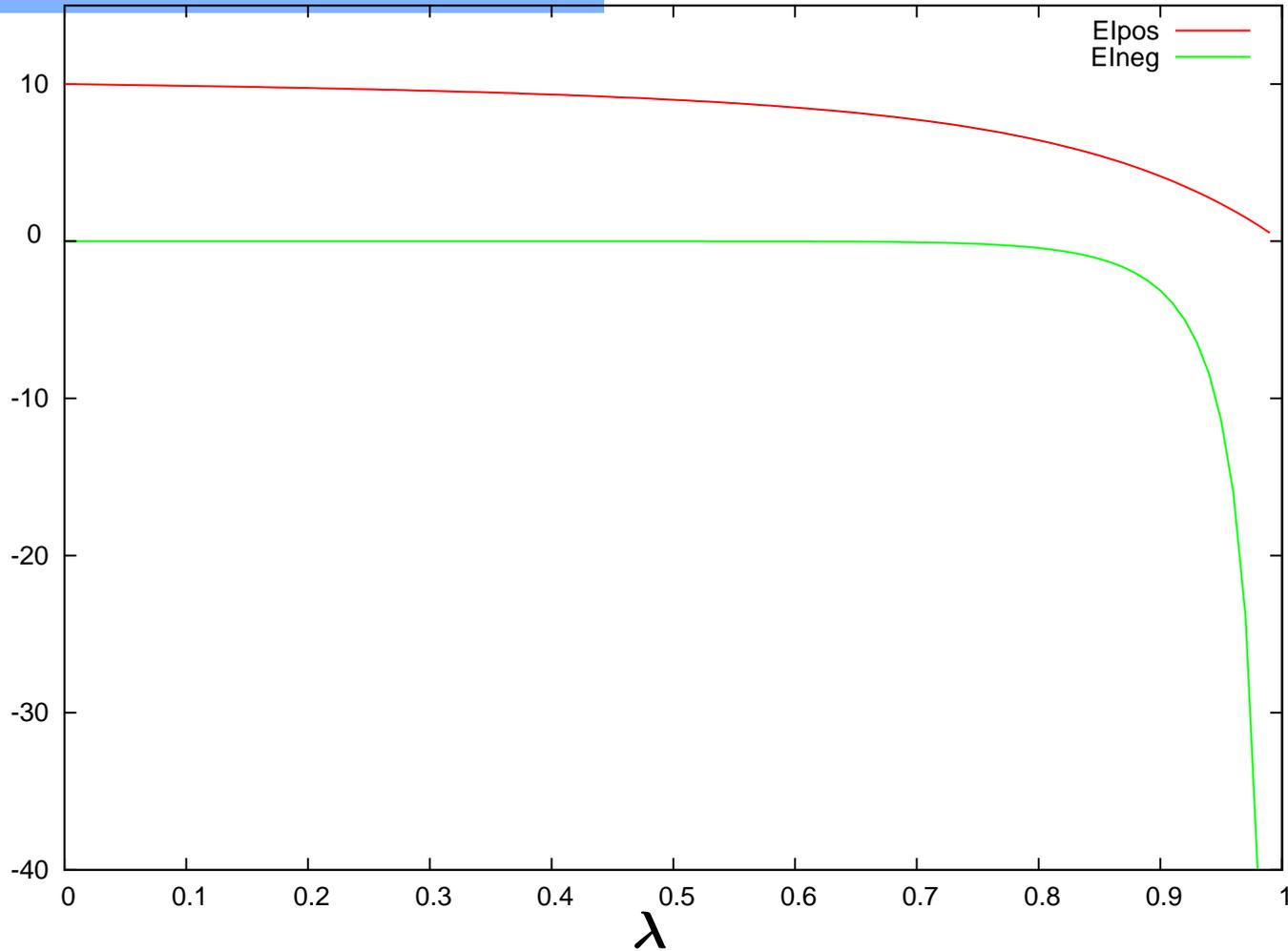
Make-to-Stock Queue

prob I pos

λ

$S = 10, \mu = 1$

Base Stock Policy

$S = 10, \mu = 1$

Base Stock Policy

prob I pos, lambda=.6
prob I pos, lambda=.9

$\lambda = .6, .9, \mu = 1$

Base Stock Policy

Legend:
- EIpos, lambda=.6
- EIneg, lambda=.6
- EIpos, lambda=.9
- EIneg, lambda=.9

$\lambda = .6, .9, \mu = 1$

Base Stock Policy

We have analyzed a policy but we haven't completely specified it. *How do we select S?*

One possible way: choose S to minimize a function that penalizes expected costs due to finished goods inventory ($E(I^+)$) and backlog ($E(I^-)$). That is, minimize

$$EC(I) = E(hI^+ - bI^-) = hE(I^+) - bE(I^-),$$

where h is the cost of holding one unit of inventory for one time unit and b is the cost of having one unit of backlog for one time unit.
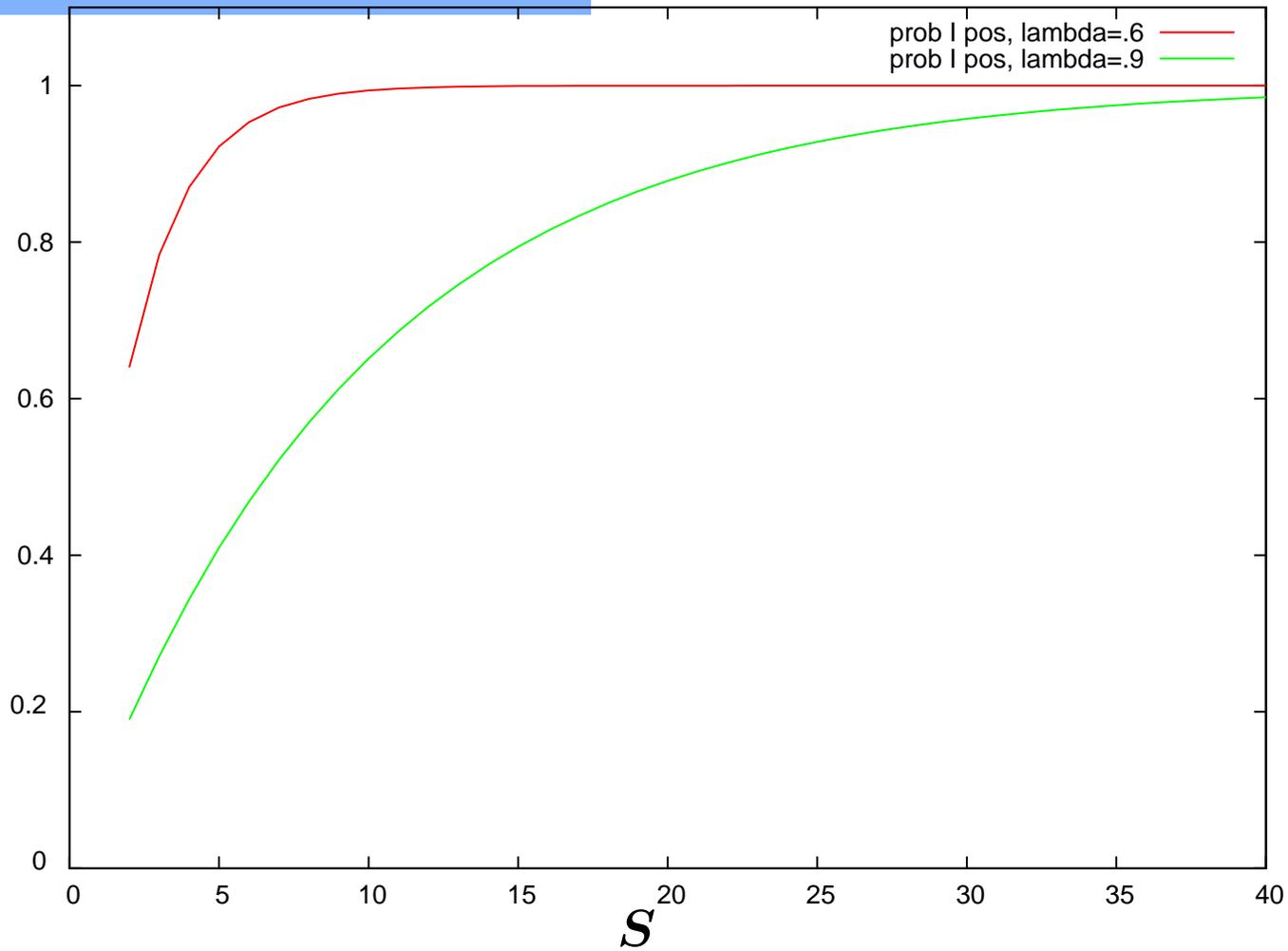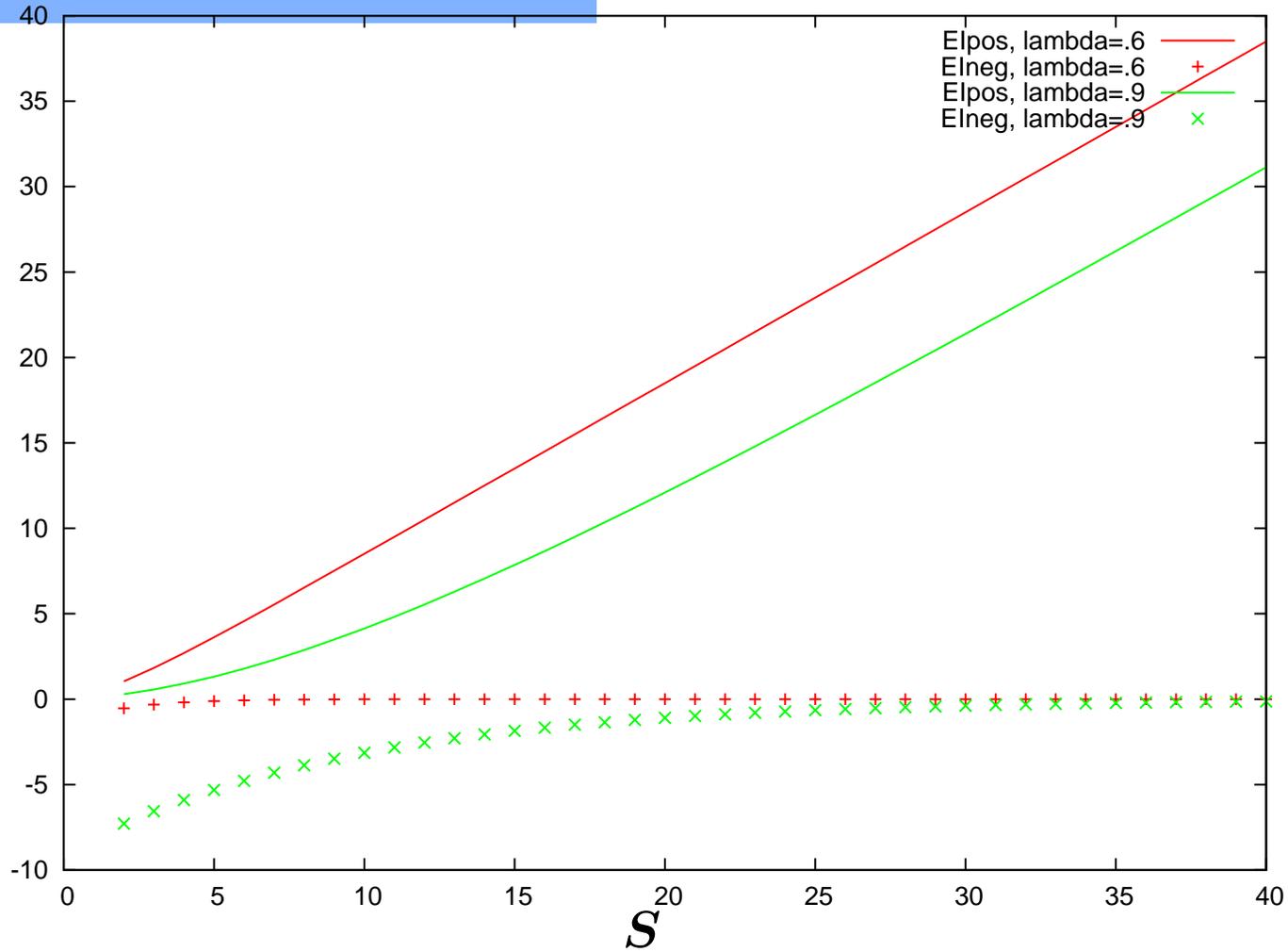
Base Stock Policy

Base Stock Policy

Make-to-Stock Queue

Optimization of Stock
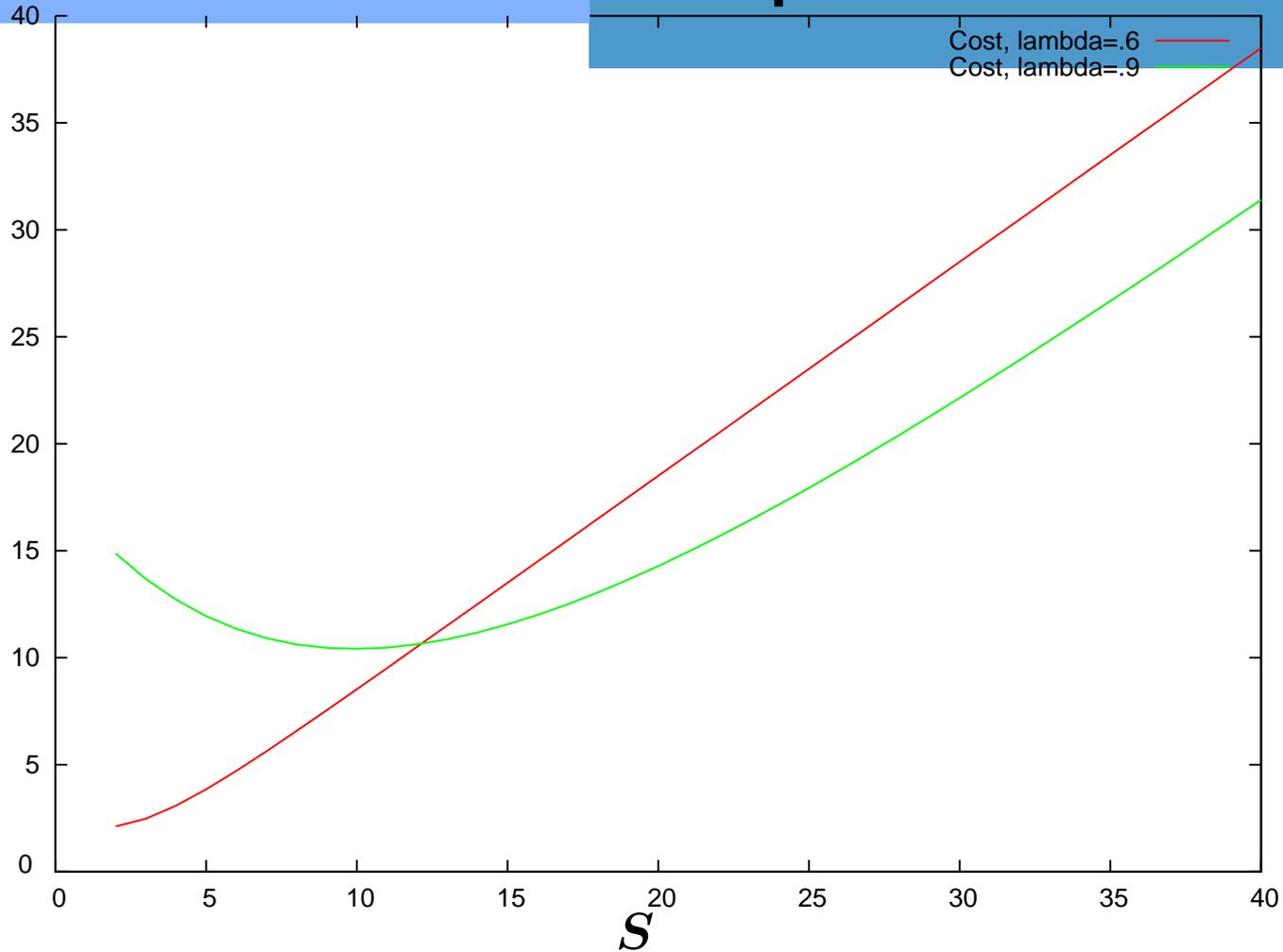
$\lambda = .6, .9, \mu = 1, h = 1, b = 2,$

Expected cost$=hE(I^+) - bE(I^-)$

Base Stock Policy

Problem: Find S^* to minimize $hE(I^+) - bE(I^-)$.

Solution[1]: $S^* = \lfloor \tilde{S} \rfloor$ or $\lfloor \tilde{S} \rfloor + 1$, where

$$\rho^{\tilde{S}+1} = \frac{h}{h+b}$$

Note: $\rho^{\tilde{S}+1} = \text{prob}(Q \geq \tilde{S} + 1) = \text{prob}(I < 0)$

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[1]Worked out by Fernando Tubilla.

That is, we choose S so that

$$\text{prob}(I < 0) = \frac{h}{h + b}$$

Look familiar? This is the solution to the newsvendor problem if $I = x - w$, $h = c - s$, and $b = r - c$ because

$$F(x^*) = \text{prob}(w \leq x^*) = \text{prob}(I \geq 0) = \frac{r - c}{(r - c) + (c - s)}$$

so

$$\text{prob}(I < 0) = 1 - \frac{r - c}{(r - c) + (c - s)} = \frac{c - s}{(r - c) + (c - s)}$$

$c - s$ is the cost of one more unit of inventory when $x > w$, ie $I > 0$.

$r - c$ is the cost of one more unit of inventory when $x < w$, ie $I < 0$.

Base Stock Policy

Optimization of Stock

profit

$r-c$

$s-c$

w

x

Newsvendor profit function

Base Stock Policy

Optimization of Stock

Questions:

- How does the problem change if we consider factory floor inventory? How does the solution change?

- How does the problem change if we consider finished goods inventory *space* ? How does the solution change?

- How else could we have selected S?

Q, R Policy

- Fixed ordering cost, like in EOQ problem
- Random demand, like in newsvendor or base stock problems.
- Policy: when the inventory level goes down to R, buy a quantity Q.
- Hard to get optimal R and Q; use EOQ and base stock ideas.

Inventory Position
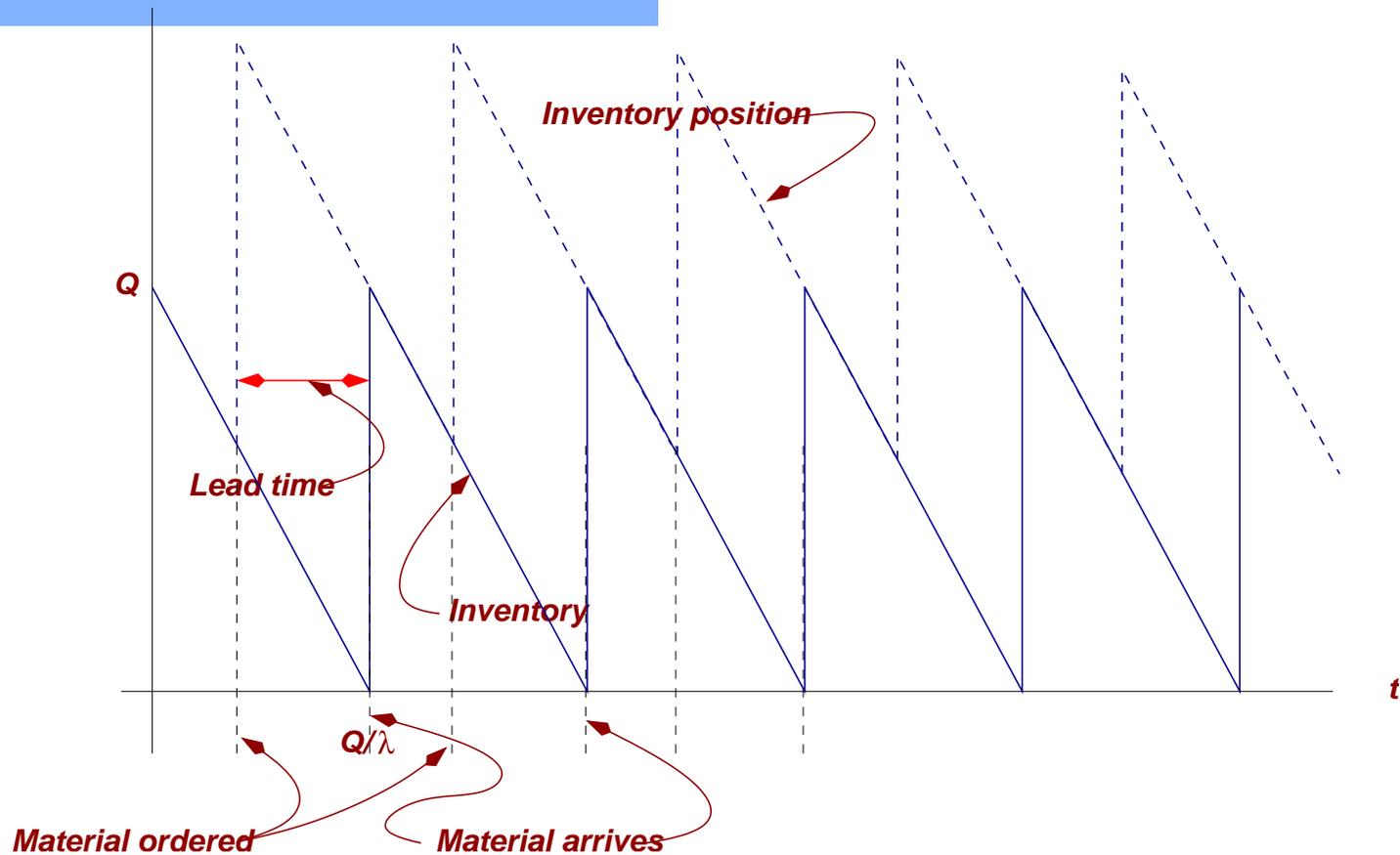
The *current* order must satisfy demand until the *next* order arrives.

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