# 22.01 Fall 2015, Problem Set 3 Solutions (Analytical Version)

Due: October 7, 11:59PM on the 22.01 Learning Modules website

October 11, 2015

Complete all the assigned problems, and do make sure to show your intermediate work. Please upload your full problem set in PDF form on the Stellar site. Make sure to upload your work at least 15 minutes early, to account for computer/network issues.

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## 1 Smoke Detector Operation

In these problems, consider the decay of $^{241}$Am, the isotope used in ionization chamber-type smoke detectors.

1. Write the two possible types of decay reactions for $^{241}$Am, and state which decay processes (and competing processes) may be possible for each general type of reaction. You don't have to address every single energy level, there are dozens! Just group them into categories.
   **Two possible decay chains are possible for** $^{241}$Am, an alpha decay with a subsequent isomeric transition (IT), and one without:
   $$^{241}Am \rightarrow^{237} Np \tag{1}$$
   $$^{241}Am \rightarrow^{237} Np^{*} + \gamma \tag{2}$$
   **No processes compete with alpha decay, though each IT (gamma decay) competes with internal conversion (IC). When IC occurs, the gamma ray may instead eject an electron with energy $E_\gamma - E_{\text{binding}}$, followed by either photon emission from a higher shell electron falling back down to this newly opened lower level, or an Auger electron.**

2. Now consider only the *three* most likely alpha decay energies of $^{241}$Am.

   (a) Draw a complete energy level diagram showing alpha decay to these three energy levels, and any possible, successive decays to the ground state.
   **The three most likely alpha decays can be found by consulting the KAERI table of nuclides at http://atom.kaeri.re.kr:8080/cgi-bin/decay?Am-241%20A. In addition, thinking ahead, the following electron level binding energies and transitions were found on the NIST X-ray transition energies database at:**
   **http://www.nist.gov/pml/data/xraytrans/index.cfm. The "K-edge" energy corresponds to the binding energy of a K-shell electron, while the K1L1 transition corresponds to a transition from the most bound L-level to the most bound K-level:. Other numbered transitions were not chosen, as they are very close to the 1-1 transitions:**

| Energy (keV) | Intensity (rel.) | Transition | Energy (keV) |
|---|---|---|---|
| 5544.5 | 0.34 | **K edge** | **118.7** |
| 5511.47 | 0.22 | **L edge** | **22.4** |
| **5485.56** | **84.5** | **K-L** | **96.2** |
| 5469.45 | 0.04 | **K-M** | **112.9** |
| **5442.80** | **13.0** | **K-N** | **117.2** |
| 5416.27 | 0.01 | **L-M** | **16.7** |
| **5388.23** | **1.6** | **L-N** | **20.9** |

**Remember that the energy of the alpha decay is not the same as the Q-value, because the $^{237}$Np recoil nucleus also takes away some of that kinetic energy released by the Q-value. One can use conservation of momentum to find out the corresponding Q-value for each alpha energy:**

$$p_\alpha = p_{Np} \tag{3}$$

$$m_\alpha v_a = m_{Np} v_{Np} \tag{4}$$

$$\sqrt{2m_\alpha E_\alpha} = \sqrt{2m_{Np} E_{Np}} \tag{5}$$

$$E_{Np} = \frac{m_\alpha}{m_{Np}} E_\alpha \tag{6}$$

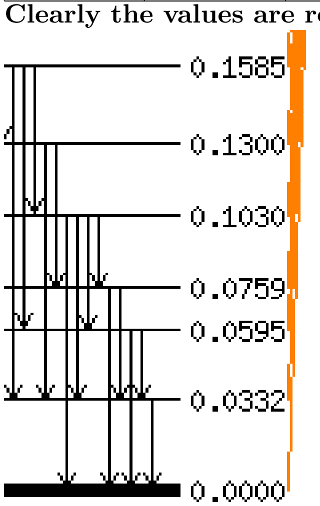
**We know the energy of the alpha particle in each case, and using this formula, we can find the corresponding Q-value for each alpha energy:**

$$Q = E_\alpha + E_{Np} = E_\alpha + \frac{m_\alpha}{m_{Np}} E_\alpha = E_\alpha \left(1 + \frac{m_\alpha}{m_{Np}}\right) = E_\alpha \left(1 + \frac{4.0026032\,amu}{237.0481673\,amu}\right) \approx 1.016885 E_\alpha \tag{7}$$

**This yields the following table of Q-values for our three alpha particle energies, and the corresponding $^{237}$Np excited states by subtracting these Q-values from the $^{241}$Am relative ground state energy (5637.8 keV):**

| $E_\alpha$ (keV) | **Q (keV)** | $^{237}$Np **Energy Level (Calc.)** | **Closest** $^{237}$Np **Level (Table)** |
|---|---|---|---|
| **5485.56** | **5578.18** | **59.62** | **59.5** |
| **5442.80** | **5534.70** | **103.1** | **103.0** |
| **5388.23** | **5479.21** | **158.59** | **158.5** |

**Clearly the values are remarkably close. The full decay diagram proceeds as follows:**




**One can see that not every energy transition is allowed. The following transitions were identified, along with their corresponding gamma ray energies. In addition, allowed electron transitions following internal conversion are tabulated for just the first four energy levels:**

| $E_i$ | $E_f$ | $\Delta E$ | **K-eject?** | **L-eject?** |
|---|---|---|---|---|
| **158.5** | **33.2** | **125.3** | $\surd$ | $\surd$ |
| **158.5** | **59.5** | **99.0** | | $\surd$ |
| **158.5** | **103.0** | **55.5** | | $\surd$ |
| **130.0** | **33.2** | **96.8** | | $\surd$ |
| **130.0** | **75.9** | **54.1** | | $\surd$ |
| **103.0** | **0** | **103.0** | | $\surd$ |
| **103.0** | **33.2** | **69.8** | | $\surd$ |
| **103.0** | **59.5** | **43.5** | | $\surd$ |
| **103.0** | **75.9** | **27.1** | | $\surd$ |
| **75.9** | **0** | **75.9** | | $\surd$ |
| **75.9** | **33.2** | **42.7** | | $\surd$ |
| **59.5** | **0** | **59.5** | | $\surd$ |
| **59.5** | **33.2** | **26.3** | | $\surd$ |
| **33.2** | **0** | **33.2** | | $\surd$ |

(b) For each of the initial alpha particle energies, separately sketch a hypothetical photon (gamma plus x-ray) spectrum that you would expect to observe. You may want to use the NIST X-Ray Transition Energy Database to help generate your answer.
**This spectrum should consist of all the gamma ray energy differences in the table above, along with all the x-ray transition energies identified, as at least one IT/IC process is energetic enough to eject a K-shell electron. The spectrum would therefore look something like this:**

3. It is clear that $^{241}$Am produces a few types of radiation at many different energies. Do you expect the alpha particles, the gamma rays, or the x-rays to be responsible for producing the largest number of ions in a fixed space (like in a smoke detector), and why?
**The alpha particles should be responsible for most of the ionizations, because they interact so much more strongly with matter, and therefore the electron clouds in the matter. Their heavy mass and high charge compared to gammas or x-rays (no mass, no charge) gives them far higher ionizing power.**

# 2 Medical Isotope Physics

In these problems, consider the decay of $^{99}$Mo, a crucial medical isotope widely used in imaging and diagnosis procedures.

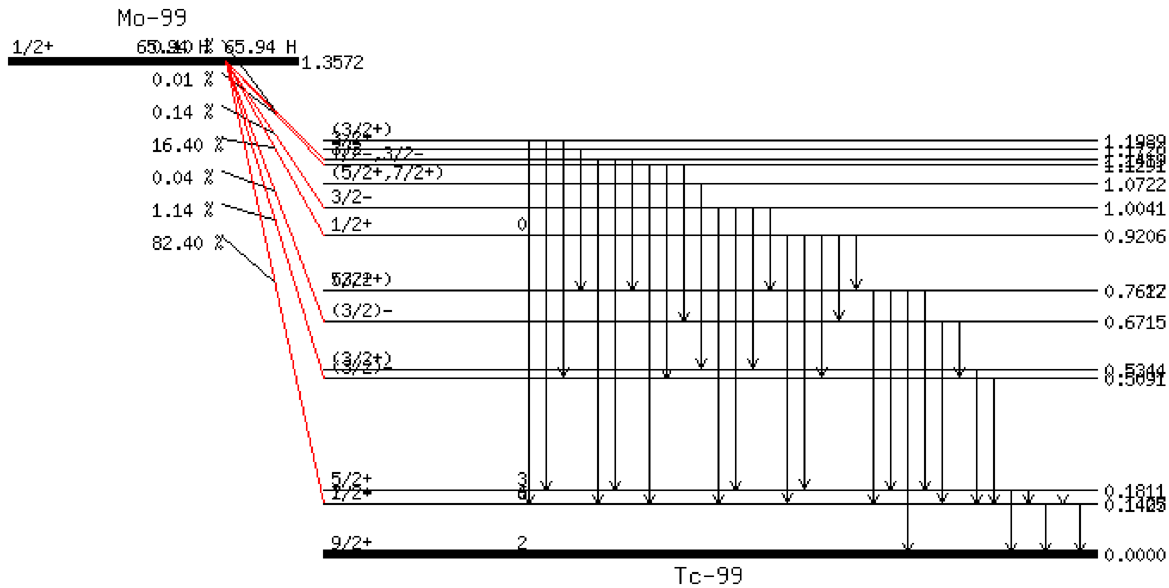
1. Calculate the Q-value for the decay of $^{99}$Mo using the binding energies of the initial and final nuclei, and any other information that you need.
The decay of $^{99}$Mo proceeds by beta decay to $^{99m}$Tc, followed by a gamma decay (IT) to stable $^{99}$Tc:

$$^{99}Mo \rightarrow ^{99m} Tc + \beta^- + \bar{\nu} ^{99m}Tc \rightarrow ^{99} Tc + \gamma \tag{8}$$

All that is required are the masses in amu of $^{99}$Mo and $^{99}$Tc, and the conversion factor between amu and MeV:

$$Q = \left( m_{Mo-99}\cancel{c^2} - m_{Tc-99}\cancel{c^2} \right) \left( \frac{931.49\, MeV}{amu - \cancel{c^2}} \right) = 931.49\, (98.9077116 - 98.9062546) = 1.3572\, MeV \tag{9}$$

This is indeed the difference in energy levels according to the decay diagram for Mo-99:




2. You may have noticed that $^{99}$Mo is an unstable isotope. Which nuclear reactions could create $^{99}$Mo? Write the nuclear reactions for these processes, and calculate their Q-values to justify your answer.
**This is where some creativity can come into play. $^{99}$Mo could either be produced by spontaneous decay reactions, or by deliberate bombardment of parent isotopes with other particles. First, the decay modes:**

$$^{103}Ru \rightarrow ^{99}Mo + \alpha; \qquad Q = (m_{Ru-103} - m_{Mo-99} - m_{\alpha})\cancel{c^2}\left(\frac{931.49\,MeV}{amu - \cancel{c^2}}\right) = -3.71\,MeV \tag{10}$$

$$^{99}Nb \rightarrow ^{99}Mo + \beta^- + \bar{\nu}; \qquad Q = 1.3572\,MeV\ (above) \tag{11}$$

$$^{99}Tc \rightarrow ^{99}Mo + \beta^+ + \nu; \qquad Q = -1.3572\,MeV \tag{12}$$

$$^{99}Tc \rightarrow ^{99}Mo + e^-\,(EC); \qquad Q = -1.3572\,MeV \tag{13}$$

$$^{100}Mo \rightarrow ^{99}Mo + n; \qquad Q = -8.29\,MeV \tag{14}$$

$$^{235}U \rightarrow ^{99}Mo + ^{134}Sn + 2^1_0n; \qquad Q = 177.4\,MeV \tag{15}$$

**Now for the energetic particle bombardment methods:**

$$^{98}Nb + p^+ \rightarrow ^{99}Mo; \qquad Q = 9.73\,MeV \tag{16}$$

$$^{98}Mo + n \rightarrow ^{99}Mo; \qquad Q = 5.93\,MeV \tag{17}$$

$$^{95}Zr + \alpha \rightarrow ^{99}Mo; \qquad Q = 2.73\,MeV \tag{18}$$

**As one can see, only beta decay and spontaneous fission is possible, though a number of bombardment options are available.**

3. For the most common $^{99}$Mo production method which does not arise from spontaneous radioactive decay, answer the following questions:

   (a) Assuming the incoming particle was at room temperature to begin with (you should find its kinetic energy), what are the possible recoil kinetic energies of the $^{99}$Mo produced?
   **The most likely and available method to produce $^{99}$Mo is by neutron capture in a reactor, as this is the most readily available source of particles already being created. The recoil question is a bit of a trick question, as only one particle ($^{99}$Mo) is created, from the merging of two other particles with about zero kinetic energy or momentum. Therefore, the $^{99}$Mo nucleus should still stay at rest, so it should have zero recoil**

**energy. This is because $^{99}$Mo is in effect a long-lived compound nucleus, which then decays by beta decay to $^{99}$Tc. The Q-value in this reaction is absorbed into creating this unstable nucleus.**
**(Not required for credit, just good to know): If one wants to explore deeper, the full reaction involves creating this excited compound nucleus of $^{99}$Mo, which then de-excites to the ground state of $^{99}$Mo via isomeric transition. One could then equate the momentum of the $^{99}$Mo nucleus and the outgoing gamma ray:**

$$\sqrt{2m_{Mo-99}E_{Mo-99}} = \frac{E_\gamma}{c}; \qquad E_{Mo-99} = \frac{E_\gamma^2}{2m_{Mo-99}c^2} \tag{19}$$

(b) How will the recoil energy and the outgoing radiation energy change if the incoming radiation had a kinetic energy of 2 MeV?
**If the neutron had an incoming kinetic energy of 2 MeV, then the momentum of the excited compound nucleus should be equal to that of the incoming neutron:**

$$\sqrt{2m_n E_n} = \sqrt{2m_{Mo-99}E_{Mo-99}}; \qquad E_{Mo-99} = \frac{m_n}{m_{Mo-99}}E_n \tag{20}$$

# 3 Q-Equation Derivation

1. Consider the kinematic system in Figure 8.1 (p. 142). In the laboratory coordinate system (LCS), show the origin of equation 8.4 (the Q-equation) by conserving energy, x-momentum, and y-momentum of all particles involved in the collision.
**The full equation describing the reaction in Figure 8.1 is as follows:**

$$i + I \rightarrow f + F + Q \tag{21}$$

**Here we must conserve the x-momentum, y-momentum, and total energy involved in the reaction:**

$$E_i + m_i c^2 + m_I c^2 = E_f + m_f c^2 + E_F + m_F c^2 \tag{22}$$

**We can therefore express Q in two different forms:**

$$Q = m_i c^2 + m_I c^2 - m_f c^2 - m_F c^2 = E_f + E_F - E_i \tag{23}$$

**Then we conserve momentum, assuming the small particle leaves at an angle $\theta$ and the large particle leaves at an angle $\phi$:**

$$m_i v_i = m_f v_f cos\theta + m_F v_F cos\phi \tag{24}$$

$$0 = m_f v_f sin\theta - m_F v_F sin\phi \tag{25}$$

**Next we move the $\theta$ and the $\phi$ terms to opposite sides of the equation (thinking ahead a bit), and substitute $p = mv = \sqrt{2mE}$ into each equation:**

$$\sqrt{2m_i E_i} - \sqrt{2m_f E_f}cos\theta = \sqrt{2m_F E_F}cos\phi \tag{26}$$

$$\sqrt{2m_f E_f}sin\theta = \sqrt{2m_F E_F}sin\phi \tag{27}$$

**Next, we cancel the $\sqrt{2}$ from every term and square each side of each equation:**

$$\left(\sqrt{m_i E_i} - \sqrt{m_f E_f}cos\theta\right)^2 = m_i E_i - 2\sqrt{m_i m_f E_i E_f}cos\theta + m_f E_f cos^2\theta = m_F^2 E_F^2 cos^2\phi \tag{28}$$

$$m_f E_f sin^2\theta = m_F E_F sin^2\phi \tag{29}$$

**Now we just add these two equations together, and recognize that $cos^2x + sin^2x = 1$:**

$$m_iE_i + m_fE_f\cancelto{1}{\left(cos^2\theta + sin^2\theta\right)} - 2\sqrt{m_im_fE_iE_f}cos\theta = m_FE_F\cancelto{1}{\left(cos^2\theta + sin^2\theta\right)} \tag{30}$$

**We recognize that $\mathrm{E_F}$ can be expressed in terms of the other particle masses and energies involved:**

$$E_F = Q + E_i - E_f \tag{31}$$

**Finally we substitute this in to Equation 30:**

$$m_iE_i + m_fE_f - 2\sqrt{m_im_fE_iE_f}cos\theta = m_F\left(Q + E_i - E_f\right) \tag{32}$$

$$\frac{m_i}{m_F}E_i + \frac{m_f}{m_F}E_f - \frac{2}{m_F}\sqrt{m_im_fE_iE_f}cos\theta = Q + E_i - E_f \tag{33}$$

$$Q = \frac{m_i}{m_F}E_i - E_i + \frac{m_f}{m_F}E_f + E_f - \frac{2}{m_F}\sqrt{m_im_fE_iE_f}cos\theta \tag{34}$$

$$Q = E_i\left(\frac{m_i}{m_F} - 1\right) + E_f\left(\frac{m_f}{m_F} + 1\right) - \frac{2}{m_F}\sqrt{m_im_fE_iE_f}cos\theta \tag{35}$$

$$Q = E_f\left(\frac{m_f}{m_F} + 1\right) - E_i\left(1 - \frac{m_i}{m_F}\right) - \frac{2}{m_F}\sqrt{m_im_fE_iE_f}cos\theta \tag{36}$$

2. How does the Q-equation change when considering particles with relativistic speeds? Write the new, complete Q-equation in this case.
**In Yip, p. 144, he states that one can simply replace the rest mass of each particle by its relativistic effective mass:**

$$m_i = m_{i_0} + \frac{E_i}{2c^2} \tag{37}$$

**This gives the following form for the Q-equation:**

$$Q = E_f\left(\frac{m_f + \frac{E_f}{2c^2}}{m_F + \frac{E_F}{2c^2}} + 1\right) - E_i\left(1 - \frac{m_i + \frac{E_i}{2c^2}}{m_F + \frac{E_F}{2c^2}}\right) - \frac{2}{m_F + \frac{E_F}{2c^2}}\sqrt{\left(m_{i_0} + \frac{E_i}{2c^2}\right)\left(m_f + \frac{E_f}{2c^2}\right)E_iE_fcos\theta} \tag{38}$$

# 4 Allowable Nuclear Reactions

For these problems, determine whether the following reactions would be allowed, and answer the additional questions.

1. Which of the following decay methods are energetically allowable from the ground state of $^{216}$At? Back up your reasoning with an energetic argument.

   (a) Alpha decay

   $$^{216}\mathrm{At} \rightarrow ^{212}Bi + \alpha; \qquad Q = 7.949\,MeV \tag{39}$$

   **Allowed, plus we get exactly the total alpha decay energy in the KAERI table.**

   (b) Beta decay

   $$^{216}\mathrm{At} \rightarrow ^{216}Rn + \beta^- + \bar{\nu}; \qquad Q = 2.003\,MeV \tag{40}$$

   **Allowed, plus we get exactly the total beta decay energy in the KAERI table.**

   (c) Positron decay

   $$^{216}\mathrm{At} \rightarrow ^{216}Po + \beta^+ + \nu; \qquad Q = 0.469\,MeV \tag{41}$$

   **Not allowed, because the Q-value isn't large enough to create the positron (1.022 MeV).**

(d) Electron capture

$$^{216}\text{At} \rightarrow ^{216} Po + \nu; \qquad Q = 0.469\, MeV \tag{42}$$

**Allowed, plus we get exactly the total electron capture decay energy in the KAERI table.**

(e) Isomeric transition

$$^{216}\text{At} \rightarrow ^{216} At; \qquad Q = 0 \tag{43}$$

**Impossible, because $^{216}$At is already at its ground state.**

(f) Spontaneous fission

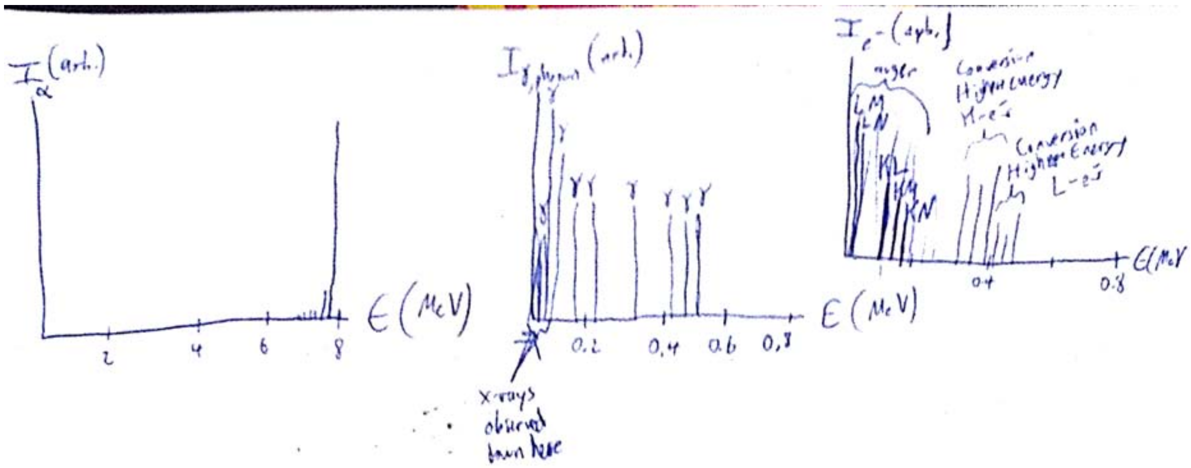
i. Can you find an instance where this particular one is energetically allowable?
**Many of them are, here is one example:**

$$^{216}\text{At} \rightarrow ^{114}_{45} Rh + ^{100}_{40} Zr + 2^{1}_{0}n; \qquad Q = 138.3\, MeV \tag{44}$$

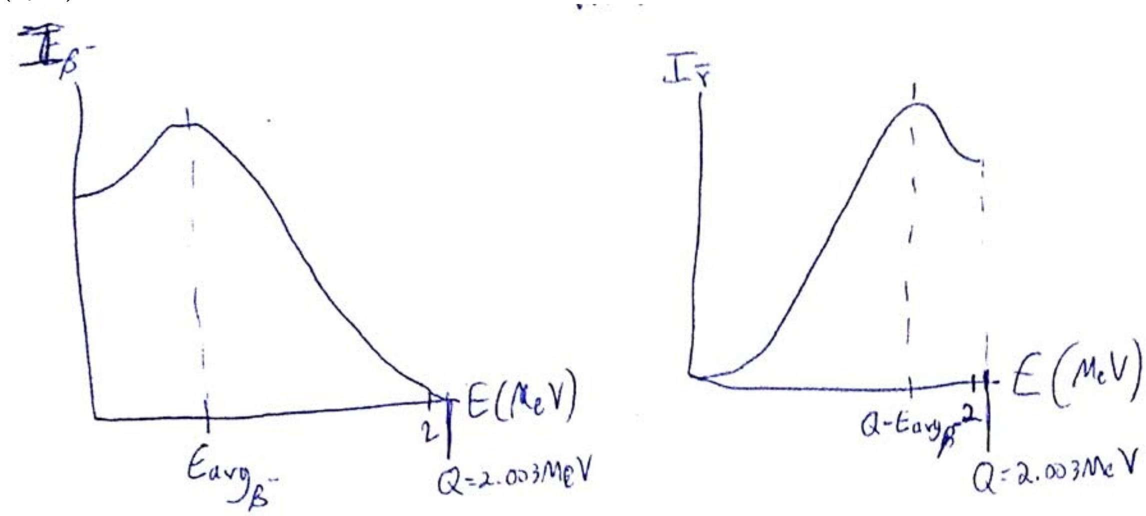
ii. Why do you think it's never observed?
**Just because a reaction is energetically allowable does not mean that it will happen. In order for a nucleus to spontaneously fizz, it must overcome a significant strong nuclear force attractive barrier. Think of it as a huge activation energy, required to release a huge amount of energy at the end of the reaction. The higher this barrier, the less likely a particular nuclear vibration will allow for the fission products to separate themselves from the nucleus. These barriers are typically on the order of 200 MeV, which is why only the super-heavy elements undergo spontaneous fission with any measurable probability.**

2. For the reactions which are allowed, write the full nuclear reaction in each case, and draw a graph of the energy spectrum you would expect to see from each released form of radiation, including secondary ejections of particles or photons.
**See above for the full nuclear reactions.**

**For alpha decay, one would expect to see mono-energetic alpha particles at the following tabulated energies: 7802 keV, 7683 keV, 7610 keV, 7560 keV, 7470 keV, 7390 keV, 7317 keV, 7240 keV. The diagram does not state any allowable gamma decays, though in reality they are likely to be observed. The short half-life (0.3 ms) of $^{216}$At is likely to blame for this lack of measured data. Therefore, without knowing the selection rules for gamma emission from spin and parity states (to be learned in 22.02), all that one can say is that the maximum energy available to eject an electron is that of the largest possible transition (573 keV). Using the NIST x-ray transition energy tables for $^{216}$At, the highest energy gamma ray could only K-shell or L-shell electrons (binding energies of 95.7 keV and 17.5 keV, respectively). Therefore, other photon transitions of KL, KM, KN, LM, or LN transitions may be observed from IC processes competing with gamma emission. One would also expect to observe Auger electrons at these energies, in addition to the K, L, M, N... conversion electrons. See the diagrams below:**




**For beta decay, one would expect to see an electron spectrum as below, with $E_{max}$ at the Q-value of 2.003 MeV. The antineutrino spectrum must be its mirror image, because the probability of observing a beta with energy E is equal to observing an antineutrino with (Q-E).**




**For electron capture, one would only expect to see the transition x-rays and corresponding Auger electrons as seen in alpha decay, maxing out at 469 keV.**

**For spontaneous fission, one would expect to see a huge range of photons, conversion electrons, transition x-rays, and whatever decay products are possible from the fission products. Then again, even though it's energetically allowable, it's improbable.**

3. For the reactions which are not allowed, under which conditions could they be allowable? In other words, how would you insert energy into the system to make them allowed, and how much?
For all reactions that are not allowed, increasing the kinetic energy of the $^{216}$At **nucleus to equal -Q (most reactions) or such that $\mathbf{Q + E_{At} \geq 1.022}$ MeV (positron) would make this allowable. This is because a *necessary and sufficient* condition for a nuclear reaction to proceed is that the sum of Q and the kinetic energy are greater than zero:**

$$Q + E_i \geq \begin{cases} 0 & (most\,reactions) \\ 1.022\,MeV & (positron\,emission) \end{cases} \tag{45}$$