22.01 Fall 2015, Quiz 1 Study Sheet

October 11, 2015

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1 Basics of Physics, Kinematics, and Relativity

$$p = mv = \sqrt{2mE}; \qquad E = \frac{1}{2}mv^2 \tag{1}$$

where p is momentum, m is mass, v is velocity, and E is kinetic energy of a particle.

$$E_\gamma = \frac{hc}{\lambda} \tag{2}$$

where h is Planck's constant, c is the speed of light, and λ is the wavelength of the photon.

$$\frac{1}{\lambda_{transition}} = Ry\left(\frac{1}{n_{final}^2} - \frac{1}{n_{initial}^2}\right) \tag{3}$$

where n corresponds to the initial and final electron shell levels, depending on the subscript. The Rydberg energy is given as follows:

$$Ry_Z = \frac{Z^2 m_{e^-} e_c^4}{8\epsilon_0^2 h^3 c} \tag{4}$$

where Z is the number of protons in the nucleus, $\mathrm{m_{e^-}}$ is the rest mass of the electron (511 keV), $\mathrm{e_c}$ is the charge on the electron (1.6×10^{-19}C), and ϵ_0 is the permittivity of a vacuum to allow electric field lines through it.

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_{relativistic}}{m_0} \tag{5}$$

where γ represents the gamma factor for relativistic motion, and $\mathrm{m_0}$ is the rest mass of the particle. It is used to compute the relativistic mass of a particle traveling at significant fractions (1% and higher) of the speed of light:

$$E_{total} = m_0\gamma c^2; \qquad E_{kinetic} = (\gamma - 1)\, m_0 c^2 \tag{6}$$

Consider the limiting cases here. If a particle is at rest (v=0, $\gamma = 1$), then its kinetic energy is zero, and its mass is equal to its rest mass. If the particle is travelling at the speed of light, then $\gamma \to \infty$ and it becomes infinitely massive. It also takes an infinite amount of kinetic energy to get a particle with non-zero mass moving at the speed of light.

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2 Nuclear Reactions and Energetics

A general nuclear reaction proceeds, and is written as follows:

$$i + I \to f + F + Q; \qquad I\,(i, f)\,F \tag{7}$$

where i and I represent the small and large initial particles, respectively, f and F represent the small and large final particles, respectively (which may not be the same ones), and Q is the energy consumed or liberated from the reaction. The last term shown is the shorthand form.

This Q value is expressible in terms of many things, stemming from conservation of total rest mass energy and kinetic energy of the reaction:

$$E_i + m_i c^2 + m_I c^2 = E_f + m_f c^2 + E_F + m_F c^2 \tag{8}$$

Separating the two sets of masses and energies to one side or the other of the equation, it can be written as follows:

$$m_i c^2 + m_I c^2 - m_f c^2 - m_F c^2 = E_f + E_F - E_i = Q \tag{9}$$

where it is assumed that the large, initial particle is at rest, unless we're in CERN or the large hadron collider or something.

The binding energy of a nucleus is analogous to the *work of separation* of its constituent nucleons, and can therefore be written as the difference between the masses of its individual nucleons and the assembled nucleus:

$$B.E.\,(A,Z) = ZM_p + (A-Z)\,M_n - M\,(A,Z) \tag{10}$$

where BE is the binding energy, Z is the proton number, A is the total number of nucleons, $\mathrm{M_p}$ is the rest mass of the proton, $\mathrm{M_n}$ is the rest mass of the neutron, and M(A,Z) is the mass of the nucleus.

All masses and energies can be equivalently expressed in units of energy, such as keV or MeV. To convert between the two, use the following conversion factor:

$$M\ [MeV] = M\ [amu] \times \left[\frac{931.49\,MeV}{amu - c^2}\right] c^2 \tag{11}$$

Pro tip: Don't round masses in amu! All those digits really count.

The excess mass is the difference in amu between the number of nucleons in a nucleus and its actual mass:

$$\Delta = A - M\,(A,Z) \tag{12}$$

Note how the excess mass and the binding energy are *directly* related:

$$B.E.\,(A,Z) = ZM_p + (A-Z)\,M_n - A + \Delta \tag{13}$$

A semi-empirical estimate of the mass of a nucleus can be found using the *liquid drop model* of the nucleus:

$$B.E.\,(A,Z) = a_v A - a_s A^{2/3} - a_c \frac{Z\,(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + a_p \delta \tag{14}$$

For definitions of the terms, see the Yip book, p. 59, equation 4.10 and the following explanation.

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3 Radioactive Decay

Spontaneous radioactive decay implies that Q>0, or that the reaction is exothermic. The opposite case would be when the reaction is *endothermic*, or would consume energy. The latter case requires additional energy to be imparted into the system to make the reaction move forward, just like in chemistry.

Spontaneous radioactive decay can proceed via a number of mechanisms, including:

3.1 Alpha (α) Decay

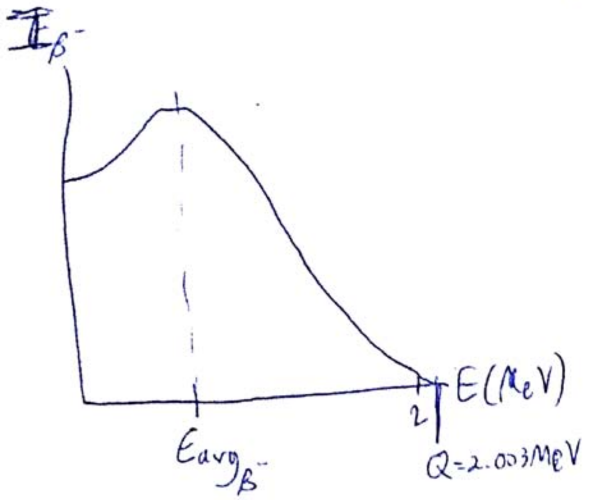
$${}^{A}_{Z}P \rightarrow {}^{A-4}_{Z-2}D + \alpha; \qquad Q\,[amu] = (m_P - m_D - m_\alpha) \tag{15}$$

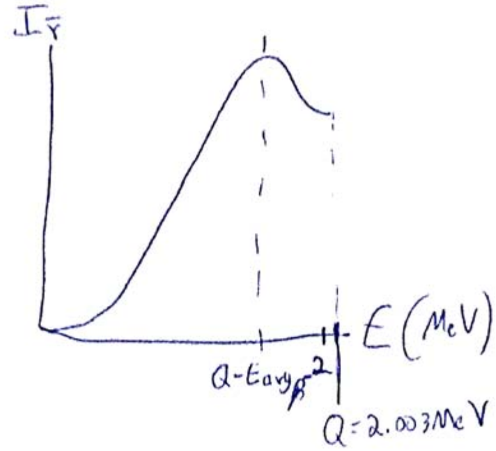
Alpha particles are emitted monoenergetically, according to allowed transitions. Alpha decay may proceed to an excited state, which would allow further isomeric transitions (IT) or internal conversions (IC).

3.2 Beta (β^-) Decay

$$ {}^{A}_{Z}P \rightarrow {}^{A}_{Z+1}D + \beta^- + \bar{\nu}; \qquad Q\,[amu] = (m_P - m_D) \tag{16} $$

Betas and associated antineutrinos are emitted with a continuous spectrum, each having an average and maximum energy:





Beta decay may proceed to an excited state, which would allow further isomeric transitions (IT) or internal conversions (IC).

3.3 Positron (β^+) Decay

$$ {}^{A}_{Z}P \rightarrow {}^{A}_{Z-1}D + \beta^+ + \nu; \qquad Q\,[amu] = (m_P - m_D) \tag{17} $$

Positrons and associated neutrinos are emitted with a continuous spectrum, each having an average and maximum energy as above, though the intensity of positrons with zero energy begins at zero. Beta decay may proceed to an excited state, which would allow further isomeric transitions (IT) or internal conversions (IC). **Q must be above 1.022 MeV for this reaction to be allowable.**

3.4 Electron Capture (EC)

Instead of emitting a positron, the nucleus may capture an inner-shell electron, binding it with a proton to create a neutron. The inner-shell hole is then plugged by higher-energy electrons falling down in energy levels, emitting characteristic photons according to Equation 3. These also compete with the emission of Auger electrons, which may be ejected from outer shells.

3.5 Isomeric Transition (IT, γ Decay)

A nucleus in an excited state may decay by gamma ray emission to a lower energy state, which may or may not be the ground state:

$$ {}^{A}_{Z}P^* \rightarrow {}^{A}_{Z}P + \gamma; \qquad Q\,[MeV] = E_\gamma \tag{18} $$

3.6 Internal Conversion (IC)

This process competes with IT, and involves the ejection of an inner-shell electron with an energy of $\mathrm{E}_{\mathrm{e}^-} = \mathrm{E}_\gamma - \mathrm{E}_{\mathrm{binding}}$, with the latter given by Equation 3 with a final shell level of ∞. This can also be followed by electron shell transitions with characteristic x-rays and/or Auger electrons as above.

3.7 Spontaneous Fission (SF)

Big nuclei just blow up sometimes. Even when it is energetically possible, the Q value needs to be high enough for the two nuclear pieces to overcome the strong nuclear force barrier and tunnel out of the nucleus. Needless to say it is a low-probability reaction, though it does happen for heavier nuclei:

$$ {}^{A}_{Z}P \rightarrow FP_1 + FP_2 + \eta {}^{1}_{0}n; \qquad Q\left[amu\right] = \left(m_P - m_{FP_1} - m_{FP_1} - \eta m_n\right) \tag{19}$$

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4 Allowable Nuclear Reactions and the Q-Equation

The full equation relating Q, the masses, energies, and angles involved in a general nuclear reaction are as follows:

$$E_f\left(1+\frac{m_f}{m_F}\right) - E_i\left(1-\frac{m_i}{m_F}\right) - \frac{2}{m_F}\sqrt{m_i m_f E_i E_f} cos\theta \tag{20}$$

Reactions involving fewer particles can be simplified by setting appropriate terms to zero. There are a few important implications to this formula:

1. A necessary and sufficient condition for a reaction to proceed is that the sum of the kinetic energy of the incoming particle and the Q-value is positive:

$$E_i + Q \geq 0 \tag{21}$$

2. If a reaction is not allowed on its own (endothermic, Q<0), then there is a threshold energy required to induce it:

$$E_{threshold} = -Q\frac{m_f + m_F}{m_f + m_F - m_i} \approx -Q\frac{m_i + m_I}{m_I} \tag{22}$$

   Note that energies are *always* positive, so for a reaction to have a threshold energy, it must be endothermic (Q<0).

3. For other implications, allowed angles, and energies, see Yip, pp. 142-149.

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5 Radioactive Decay and Half Life

Activity is defined as follows:

$$A = \lambda N \tag{23}$$

where A is the activity in Bq (or Ci), λ is the decay constant in 1/s, and N is the number of decaying atoms present. Recognizing that:

$$A = -\frac{dN}{dt} \tag{24}$$

we can write:

$$\frac{dN}{dt} = -\lambda N; \qquad N = N_0 e^{-\lambda t} \tag{25}$$

The half life $\left(t_{1/2}\right)$is solved by setting the fraction $\frac{N}{N_0}$ equal to 0.5:

$$t_{1/2} = \frac{ln\left(2\right)}{\lambda} \tag{26}$$

The concentration of a particular isotope, or chain of isotopes, can always be written as the balance between production and destruction:

$$\frac{dN_i}{dt} = Prod._i - Destr._i \tag{27}$$

Production can either be directly from bombarded particles, or from another radioactive decay:

$$Prod._i = N_{i-1}\sigma_{capture_{i-1}}\Phi + \lambda_{i-1}N_{i-1} \tag{28}$$

where σ is the *cross section*, or interaction probability, of capture by a flux Φ of incoming particles, and N represents a number density of isotope *i* or *i-1*. Number densities are calculated as follows:

$$N = \frac{\rho N_{Av}}{MM} \tag{29}$$

where ρ is the density, N_{Av} is Avogadro's number, and MM is the molar mass (or molecular weight). Note that the *macroscopic cross section* Σ accounts for both the amount of isotope *i-1* present and the probability that isotope *i-1* undergoes a reaction to produce isotope *i*:

$$\Sigma = N\sigma \tag{30}$$

These can be constructed into a series of differential equations, which can be solved to obtain the concentrations of different isotopes. For example, let's say we have a quantity of isotope N_1 at N_{10}, and it decays into isotope N_2, which also decays into isotope N_3. Isotope N_2, however, also captures neutrons (that's right, we're in a reactor now) with a characteristic cross section σ_{c_2}:

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \tag{31}$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 - N_2\sigma_{c_2}\Phi_{reactor} \tag{32}$$

$$\frac{dN_3}{dt} = \lambda_2 N_2 \tag{33}$$

Things to keep in mind include:

1. Cases in which coefficients are wildly different, for example, what happens if $\lambda_1 \gg \lambda_2$ or $\lambda_1 \ll \lambda_2$?
2. Behavior during very short times
3. Finding maximum concentrations of a given isotope, by setting the derivative equal to zero