3. Scattering, Tunneling and Alpha Decay

3.1 Review: Energy eigenvalue problem

The time-independent wavefunction obeys the time-independent Schrödinger equation:

$$\mathcal{H}\varphi(\vec{x}) = E\varphi(\vec{x})$$

where E is identified as the energy of the system. If the wavefunction is given by just its time-independent part, $\psi(\vec{x}, t) = \varphi(\vec{x})$, the state is *stationary*. Thus, the time-independent Schrödinger equation allows us to find stationary states of the system, given a certain Hamiltonian.

Notice that the time-independent Schrödinger equation is nothing else than the eigenvalue equation for the Hamiltonian operator.

The energy of a particle has contributions from the kinetic energy as well as the potential energy:

$$\mathcal{H} = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + V(\hat{x}, \hat{y}, \hat{z})$$

or more explicitly:

$$\mathcal{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(x, y, z)$$

which can be written in a compact form as

$$\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x, y, z)$$

(Notice that $V(x, y, z)$ is just a multiplicative operator, in the same way as the position is).

In 1D, for a free particle there is no potential energy, but only kinetic energy that we can rewrite as:

$$\mathcal{H} = \frac{1}{2m}p^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

The eigenvalue problem $\mathcal{H}w_n(x) = E_n w_n(x)$ is then the differential equation

$$\mathcal{H}w_n(x) = E_n w_n(x) \quad \rightarrow \quad -\frac{\hbar^2}{2m}\frac{\partial^2 w_n(x)}{\partial x^2} = E_n w_n(x)$$

For a free particle there is no restriction on the possible energies, E_n can be any positive number. The solution to the eigenvalue problem is then the eigenfunction:

$$w_n(x) = A\sin(k_n x) + B\cos(k_n x) = A'e^{ik_n x} + B'e^{-ik_n x}$$

which represents two waves traveling in opposite directions.

We see that there are two independent functions for each eigenvalue E_n. Also there are two distinct momentum eigenvalues $\pm k_n$ for each energy eigenvalue, which correspond to two different directions of propagation of the wave function $e^{\pm ik_n x}$.

3.2 Unbound Problems in Quantum Mechanics

We will then solve the time-independent Schrödinger equation in some interesting 1D cases that relate to scattering problems.

3.2.1 Infinite barrier

We first consider a potential as in Fig. 14. We consider two cases:
- Case A. The system (a particle) has a total energy larger than the potential barrier $E > V_H$.
- Case B. The energy is smaller than the potential barrier, $E < V_H$.
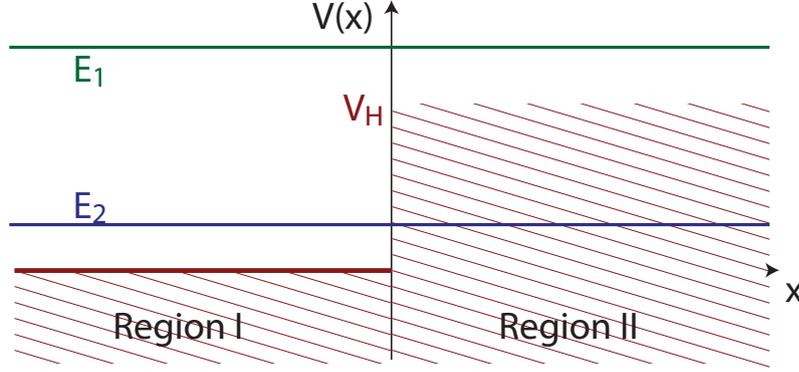

Fig. 14: Potential function and total energy of the particle

Let's first consider the classical problem. The system is a rigid ball with total energy E given by the sum of the kinetic and potential energy. If we keep the total energy fixed, the kinetic energies are different in the two regions:

$$T_I = E \qquad T_{II} = E - V_H$$

If $E > V_H$, the kinetic energy in region two is $T_{II} = \frac{p^2}{2m} = E - V_H$, yielding simply a reduced velocity for the particle. If $E < V_H$ instead, we would obtain a negative T_{II} kinetic energy. This is not an allowed solution, but it means that the particle cannot travel into Region II and it's instead confined in Region I: The particle bounces off the potential barrier.
In quantum mechanics we need to solve the Schrödinger equation in order to find the wavefunction describing the particle at any position. The time-independent Schrödinger equation is

$$\mathcal{H}\psi(x) = -\frac{\hbar^2 d^2}{2m dx^2}\psi(x) + V(x)\psi(x) = E\psi(x) \quad \rightarrow \quad \begin{cases} -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\,\psi(x) \ \text{ in Region I} \\ -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = (E - V_H)\psi(x) \ \text{ in Region II} \end{cases}$$

The two cases differ because in Region II the energy difference $\Delta E = E - V_H$ is either positive or negative.

A. Positive energy

Let's first consider the case in which $\Delta E = E - V_H > 0$. In both regions the particle behaves as a free particle with energy $E_I = E$ and $E_{II} = E - V_H$. We have already seen the solutions to such differential equation. These are:

$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II}(x) = Ce^{ik'x} + De^{-ik'x}$$

where $\frac{\hbar^2 k^2}{2m} = E$ and $\frac{\hbar^2 k'^2}{2m} = E - V_H$.
We already interpreted the function e^{ikx} as a wave traveling from left to right and e^{-ikx} as a wave traveling from right to left. We then consider a case similar to the classical case, in which a ball was sent toward a barrier. Then the particle is initially described as a wave traveling from left to right in Region I. At the potential barrier the particle can either be reflected, giving rise to a wave traveling from right to left in Region I, or be transmitted, yielding a

wave traveling from left to right in Region II. This solution is described by the equations above if we set $D = 0$, implying that there is no wave originating from the far right.

Since the wavefunction should describe a physical situation, we want it to be a continuous function and with continuous derivative. Thus we have to match the solution values and their derivatives at the boundary $x = 0$. This will give equations for the coefficients, allowing us to find the exact solution of the Schrödinger equation. This is a *boundary conditions* problem.

From

$$\psi_I(0) = \psi_{II}(0) \ \text{ and } \ \psi'_I(0) = \psi'_{II}(0)$$

and $D = 0$ we obtain the conditions:

$$A + B = C, \qquad ik(A - B) = ik'C$$

with solutions

$$B = \frac{k - k'}{k + k'}A, \qquad C = \frac{2k}{k + k'}A$$

We can further find A by interpreting the wavefunction in terms of a flux of particles. We thus fix the incoming wave flux to be Γ which sets $|A| = \sqrt{\frac{m\Gamma}{\hbar k}}$ (we can consider A to be a real, positive number for simplicity). Then we have:

$$B = \frac{k - k'}{k + k'}\sqrt{\frac{m\Gamma}{\hbar k}}, \qquad C = \frac{2k}{k + k'}\sqrt{\frac{m\Gamma}{\hbar k}}$$

We can also verify the following identity

$$k|A|^2 = k|B|^2 + k'|C|^2$$

which follows from:

$$k|B|^2 + k'|C|^2 = \frac{|A|^2}{(k + k')^2}[k(k - k')^2 + k'(2k)^2] = k|A|^2\frac{(k - k')^2 + 4k'k}{(k + k')^2}$$

Let us multiply it by $\hbar/m$:

$$\frac{\hbar k}{m}|A|^2 = \frac{\hbar k}{m}|B|^2 + \frac{\hbar k'}{m}|C|^2$$

Recall (page 26) the interpretation of $\psi(x) = Ae^{ikx}$ as a wave giving a flux of particles $|\psi(x)|^2 v = |A|^2\frac{\hbar k}{m}$. This relationship similarly holds for the flux in region II as well as for the reflected flux. Then we can interpret the equality above as an equality of particle flux:

The incoming flux $\Gamma = \frac{\hbar k}{m}|A|^2$ is equal to the sum of the reflected $\Gamma_R = \frac{\hbar k}{m}|B|^2$ and transmitted $\Gamma_T = \frac{\hbar k}{m}|C|^2$ fluxes. The particle flux is conserved. We can then define the reflection and transmission coefficients as:

$$\Gamma = \Gamma_R + \Gamma_T = R\Gamma + T\Gamma$$

where

$$R = \frac{k|B|^2}{k|A|^2} = \left(\frac{k - k'}{k + k'}\right)^2, \quad T = \frac{k'|C|^2}{k|A|^2} = \left(\frac{2k}{k + k'}\right)^2\frac{k'}{k}$$

It's then easy to see that $T + R = 1$ and we can interpret the reflection and transmission coefficients as the reflection and transmission probability, respectively.

In line with the probabilistic nature of quantum mechanics, we see that the solution of the Schrödinger equation does not give us a precise location for the particle. Instead it describes the probability of finding the particle at any point in space. Given the wavefunction found above we can then calculate various quantity of interest, such as the probability of the particle having a given momentum, position and energy.

B. Negative energy

Now we turn to the case where $E < V_H$, so that $\Delta E < 0$. In the classical case we saw that this implied the impossibility for the ball to be in region II. In quantum mechanics we cannot simply guess a solution based on our intuition, but we need again to solve the Schrödinger equation. The only difference is that now in region II we have $\frac{\hbar^2 k''^2}{2m} = E - V_H < 0$.

As quantum mechanics is defined in a complex space, this does not pose any problem (we can have negative kinetic energies even if the total energy is positive) and we can solve for k'' simply finding an imaginary number $k'' = i\kappa$, $\kappa = \sqrt{\frac{2m}{\hbar^2}(V_H - E)}$ (with κ real).

The solutions to the eigenvalue problem are similar to what already seen:

$$\psi_I(x) = Ae^{ikx} + Be^{-ikx}$$

$$\psi_{II}(x) = Ce^{ik''x} = Ce^{-\kappa x},$$

where we took $D = 0$ as before.

Quantum mechanics allows the particle to enter the classical forbidden region, but the wavefunction becomes a vanishing exponential function. This means that even if the particle can indeed enter the forbidden region, it cannot go very far, the probability of finding the particle far away from the potential barrier (given by $P(x > 0) = |\psi_{II}(x)|^2 = |C|^2 e^{-2\kappa x}$) becomes smaller and smaller.

Again we match the function and its derivatives at the boundary to find the coefficients:

$$\psi_I(0) = \psi_{II}(0) \quad \rightarrow \quad A + B = C$$

$$\psi'_I(0) = \psi'_{II}(0) \quad \rightarrow \quad ik(A - B) = -\kappa C$$

with solutions

$$B = \frac{k - i\kappa}{k + i\kappa} A, \quad C = \frac{2k}{k + i\kappa} A$$

The situation in terms of flux is instead quite different. We now have the equality: $k|B|^2 = k|A|^2$:

$$k|B|^2 = k \left| \frac{k - i\kappa}{k + i\kappa} \right|^2 = k \frac{k^2 + \kappa^2}{k^2 + \kappa^2} = k.$$

In terms of flux, we can write this relationship as $\Gamma = \Gamma_R$, which implies $R = 1$ and $T = 0$. Thus we have no transmission, just perfect reflection, although there is a penetration of the probability in the forbidden region. This can be called an *evanescent* transmitted wave.

3.2.2 Finite barrier

We now consider a different potential which creates a finite barrier of height V_H between $x = 0$ and L. As depicted in Fig. 15 this potential divides the space in 3 regions. Again we consider two cases, where the total energy of the particle is greater or smaller than V_H. Classically, we consider a ball initially in Region I. Then in the case where
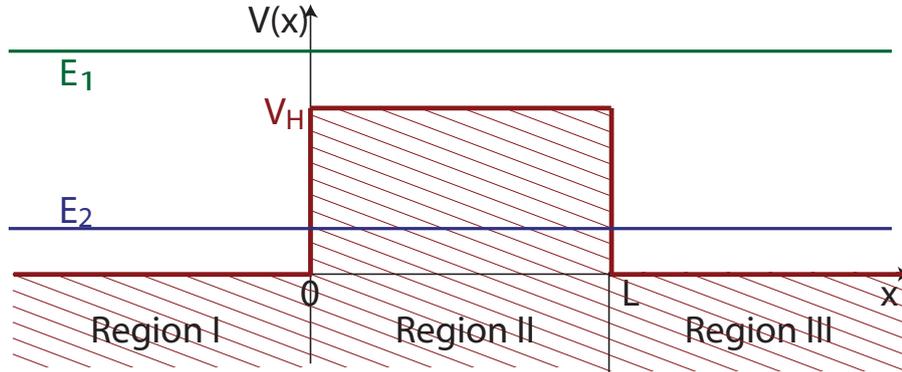

Fig. 15: Finite barrier potential

$E > V_H$ the ball can travel everywhere, in all the three regions, while for $E < V_H$ it is going to be confined in Region I, and we have perfect reflection. We will consider now the quantum mechanical case.

A. Positive kinetic energy

First we consider the case where $\Delta E = E - V_H > 0$. The kinetic energies in the three regions are

Region I	Region II	Region III
$T = \frac{\hbar^2 k^2}{2m} = E$	$T = \frac{\hbar^2 k'^2}{2m} = E - V_H$	$T = \frac{\hbar^2 k^2}{2m} = E$

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And the wavefunction is

Region I	Region II	Region III
$Ae^{ikx} + Be^{-ikx}$	$Ce^{ik'x} + De^{-ik'x}$	Ee^{ikx}

(again we put the term $Fe^{-ikx} = 0$ for physical reasons, in analogy with the classical case studied). The coefficients can be calculated by considering the boundary conditions.

In particular, we are interested in the probability of transmission of the beam through the barrier and into region III. The transmission coefficient is then the ratio of the outgoing flux in Region III to the incoming flux in Region I (both of these fluxes travel to the Right, so we label them by R):

$$T = \frac{k|\psi_{III}^R|^2}{k|\psi_I^R|^2} = \frac{k|E|^2}{k|A|^2} = \frac{|E|^2}{|A|^2}$$

while the reflection coefficient is the ratio of the reflected (from right to left, labeled L) and incoming (from left to right, labeled R) flux in Region I:

$$R = \frac{k|\psi_I^L|^2}{k|\psi_I^R|^2} = \frac{k|B|^2}{k|A|^2} = \frac{|B|^2}{|A|^2}$$

We can solve explicitly the boundary conditions:

$$\psi_I(0) = \psi_{II}(0) \qquad \psi_{II}(L) = \psi_{III}(L)$$

$$\psi_I'(0) = \psi_{II}'(0) \qquad \psi_{II}'(L) = \psi_{III}'(L)$$

and find the coefficients B, C, D, E (A is determined from the flux intensity Γ. From the full solution we can verify that $T + R = 1$, as it should be physically.

Obs. Notice that we could also have found a different solution, e.g. in which we set $F \neq 0$ and $A = 0$, corresponding to a particle originating from the right.

B. Negative Energy

In the case where $\Delta E = E - V_H < 0$, in region II we expect as before an imaginary momentum. In fact we find

Region I	Region II	Region III
$k = \sqrt{\frac{2mE}{\hbar^2}}$	$k' = i\kappa, \ \kappa = \sqrt{\frac{2m(V_H-E)}{\hbar^2}}$	$k = \sqrt{\frac{2mE}{\hbar^2}}$

And the wavefunction is

Region I	Region II	Region III
$Ae^{ikx} + Be^{-ikx}$	$Ce^{-\kappa x} + De^{\kappa x}$	Ee^{ikx}

The difference here is that a finite transmission through the barrier is possible and the transmission coefficient is not zero. Indeed, from the full solution of the boundary condition problem, we can find as in the previous case the coefficients T and R and we have $T + R = 1$.

There is thus a probability that the particle *tunnels* through the finite barrier and appears in Region III, then continuing to $x \to \infty$.

Obs. Although we have been describing the situation in terms of wave traveling in one direction or the other, what we are describing here is not a time-dependent problem. There is no time-dependence at all in this problem (all the solutions are only a function of x, not of time). This is the same situation as stationary waves for example in a rope. The state of the system is not evolving. It is always (at any time) described by the same waves and thus at any time we will have the same outcomes and probability outcomes for any measurement.

C. Estimates and scaling

Instead of solving exactly the problem for the second case, we try to make some estimates in the case there is a very small tunneling probability. In this case we have the following approximations for the coefficients A, B, C and D.

- Assuming $T \ll 1$ we expect $D \approx 0$ since if there is a very small probability for the particle to be in region III, the probability of coming back from it through the barrier must be even smaller (in other words, if $D \neq 0$ we would have an increasing probability to have a wave coming out of the barrier).

- Also, $T \ll 1$ implies $R \approx 1$. This means that $B/A \approx 1$ or $B \approx A$.

- Matching the wavefunction at $x = 0$, we have $C = A + B \approx 2A$.
- Finally matching the wavefunction at $x = L$ we obtain:
$$\psi(L) = Ce^{-\kappa L} = 2Ae^{-\kappa L} = Ee^{ikL}$$

We can then calculate the transmission probability T from $T = \frac{k|\psi_{III}^R|^2}{k|\psi_I^R|^2}$, with these assumptions. We obtain
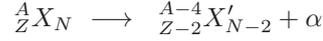
$$T = \frac{k|E|^2}{k|A|^2} = \frac{|E|^2}{|A|^2} = \frac{4|A|^2 e^{-2\kappa L}}{|A|^2} \quad \rightarrow \quad T = 4e^{-2\kappa L}$$

Thus the transmission probability depends on the length of the potential barrier (the longer the barrier the less transmission we have, as it is intuitive) and on the coefficient κ. Notice that κ depends on the difference between the particle energy and the potential strength: If the particle energy is near the edge of the potential barrier (that is, $\Delta E \approx 0$) then $\kappa \approx 0$ and there's a high probability of tunneling. This case is however against our first assumptions of small tunneling (that's why we obtain the unphysical result that $T \approx 4$!!). The case we are considering is instead where the particle energy is small compared to the potential, so that κ is large, and the particle has a very low probability of tunneling through.

3.3 Alpha decay

If we go back to the binding energy per mass number plot (B/A vs. A) we see that there is a bump (a peak) for $A \sim 60 - 100$. This means that there is a corresponding minimum (or energy optimum) around these numbers. Then the heavier nuclei will want to decay toward this lighter nuclides, by shedding some protons and neutrons. More specifically, the decrease in binding energy at high A is due to Coulomb repulsion. Coulomb repulsion grows in fact as Z^2, much faster than the nuclear force which is $\propto A$.

This could be thought as a similar process to what happens in the fission process: from a parent nuclide, two daughter nuclides are created. In the α decay we have specifically:

$$^A_Z X_N \longrightarrow \ ^{A-4}_{Z-2} X'_{N-2} + \alpha$$

where α is the nucleus of He-4: $^4_2\text{He}_2$.

The α decay should be competing with other processes, such as the fission into equal daughter nuclides, or into pairs including ^{12}C or ^{16}O that have larger B/A then α. However α decay is usually favored. In order to understand this, we start by looking at the energetic of the decay, but we will need to study the quantum origin of the decay to arrive at a full explanation.
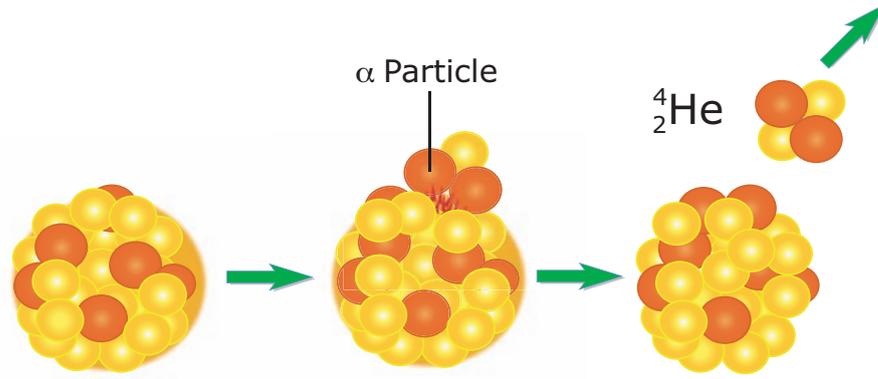

Image by MIT OpenCourseWare.

Fig. 16: Alpha decay schematics

3.3.1 Energetics

In analyzing a radioactive decay (or any nuclear reaction) an important quantity is Q, the net energy released in the decay: $Q = (m_X - m_{X'} - m_\alpha)c^2$. This is also equal to the total kinetic energy of the fragments, here $Q = T_{X'} + T_\alpha$ (here assuming that the parent nuclide is at rest).

When $Q > 0$ energy is released in the nuclear reaction, while for $Q < 0$ we need to provide energy to make the reaction happen. As in chemistry, we expect the first reaction to be a spontaneous reaction, while the second one does not happen in nature without intervention. (The first reaction is exo-energetic the second endo-energetic).

Notice that it's no coincidence that it's called Q. In practice given some reagents and products, Q give the *quality* of the reaction, i.e. how energetically favorable, hence probable, it is. For example in the alpha-decay $\log{(t_{1/2})} \propto \frac{1}{\sqrt{Q_\alpha}}$, which is the Geiger-Nuttall rule (1928).

The alpha particle carries away most of the kinetic energy (since it is much lighter) and by measuring this kinetic energy experimentally it is possible to know the masses of unstable nuclides.

We can calculate Q using the SEMF. Then:

$$Q_\alpha = B(^{A-4}_{Z-2}X'_{N-2}) + B(^4He) - B(^A_Z X_N) = B(A-4, Z-2) - B(A, Z) + B(^4He)$$

We can approximate the finite difference with the relevant gradient:

$$Q_\alpha = [B(A-4, Z-2) - B(A, Z-2)] + [B(A, Z-2) - B(A, Z)] + B(^4He) \approx = -4\frac{\partial B}{\partial A} - 2\frac{\partial B}{\partial Z} + B(^4He)$$

$$= 28.3 - 4a_v + \frac{8}{3}a_s A^{-1/3} + 4a_c \left(1 - \frac{Z}{3A}\right)\left(\frac{Z}{A^{1/3}}\right) - 4a_{sym}\left(1 - \frac{2Z}{A} + 3a_p A^{-7/4}\right)^2$$

Since we are looking at heavy nuclei, we know that $Z \approx 0.41A$ (instead of $Z \approx A/2$) and we obtain

$$Q_\alpha \approx -36.68 + 44.9A^{-1/3} + 1.02A^{2/3},$$

where the second term comes from the surface contribution and the last term is the Coulomb term (we neglect the pairing term, since a priori we do not know if a_p is zero or not).

Then, the Coulomb term, although small, makes Q increase at large A. We find that $Q \geq 0$ for $A \gtrsim 150$, and it is $Q \approx 6$MeV for $A = 200$. Although $Q > 0$, we find experimentally that α decay only arise for $A \geq 200$.

Further, take for example Francium-200 ($^{200}_{87}$Fr$_{113}$). If we calculate Q_α from the experimentally found mass differences we obtain $Q_\alpha \approx 7.6$MeV (the product is ^{196}At). We can do the same calculation for the hypothetical decay into a ^{12}C and remaining fragment ($^{188}_{81}$Tl $_{107}$):

$$Q_{^{12}C} = c^2[m(^A_Z X_N) - m(^{A-12}_{Z-6} X'_{N-6}) - m(^{12}C)] \approx 28 MeV$$

Thus this second reaction seems to be more energetic, hence more favorable than the alpha-decay, yet it does not occur (some decays involving C-12 have been observed, but their branching ratios are much smaller).

Thus, looking only at the energetic of the decay does not explain some questions that surround the alpha decay:

- Why there's no ^{12}C-decay? (or to some of this tightly bound nuclides, e.g O-16 etc.)

- Why there's no spontaneous fission into equal daughters?

- Why there's alpha decay only for $A \geq 200$?

- What is the explanation of Geiger-Nuttall rule? $\log t_{1/2} \propto \frac{1}{\sqrt{Q_\alpha}}$

3.3.2 Quantum mechanics description of alpha decay

We will use a semi-classical model (that is, combining quantum mechanics with classical physics) to answer the questions above.

In order to study the quantum mechanical process underlying alpha decay, we consider the interaction between the daughter nuclide and the alpha particle. Just prior to separation, we can consider this pair to be already present *inside* the parent nuclide, in a bound state. We will describe this pair of particles in their *center of mass* coordinate frames: thus we are interested in the relative motion (and kinetic energy) of the two particles. As often done in these situations, we can describe the relative motion of two particles as the motion of a single particle of reduced mass $\mu = \frac{m_\alpha m'}{m_\alpha + m'}$ (where m' is the mass of the daughter nuclide).

Consider for example the reaction ^{238}U→^{234}Th+α. What is the interaction between the Th and alpha particle in the bound state?

- At short distance we have the nuclear force binding the ^{238}U.

- At long distances, the coulomb interaction predominates

The nuclear force is a very strong, attractive force, while the Coulomb force among protons is repulsive and will tend to expel the alpha particle.

Since the final state is known to have an energy $Q_\alpha = 4.3$MeV, we will take this energy to be as well the initial energy of the two particles in the potential well (we assume that $Q_\alpha = E$ since Q is the kinetic energy while the potential energy is zero). The size of the potential well can be calculated as the sum of the daughter nuclide (^{234}Th) and alpha radii:

$$R = R' + R_\alpha = R_0((234)^{1/3} + 4^{1/3}) = 9.3\text{fm}.$$

On the other side, the Coulomb energy at this separation is $V_{Coul} = e^2 Z' Z_\alpha / R = 28 MeV \gg Q_\alpha$ (here $Z' = Z - 2$). Then, the particles are inside a well, with a high barrier (as $V_{Coul} \gg Q$) but there is some probability of tunneling, since $Q > 0$ and the state is not stably bound.

Thus, if the parent nuclide, ^{238}U, was really composed of an alpha-particle and of the daughter nuclide, ^{234}Th, then with some probability the system would be in a bound state and with some probability in a decayed state, with the alpha particle outside the potential barrier. This last probability can be calculated from the tunneling probability P_T we studied in the previous section, given by the amplitude square of the wavefunction outside the barrier, $P_T = |\psi(R_{out})|^2$.

How do we relate this probability to the decay rate?

We need to multiply the probability of tunneling P_T by the frequency f at which ^{238}U could actually be found as being in two fragments ^{234}Th+α (although still bound together inside the potential barrier). The decay rate is then given by $\lambda_\alpha = f P_T$.

To estimate the frequency f, we equate it with the frequency at which the compound particle in the center of mass frame is at the well boundary: $f = v_{in}/R$, where v_{in} is the velocity of the particles when they are inside the well
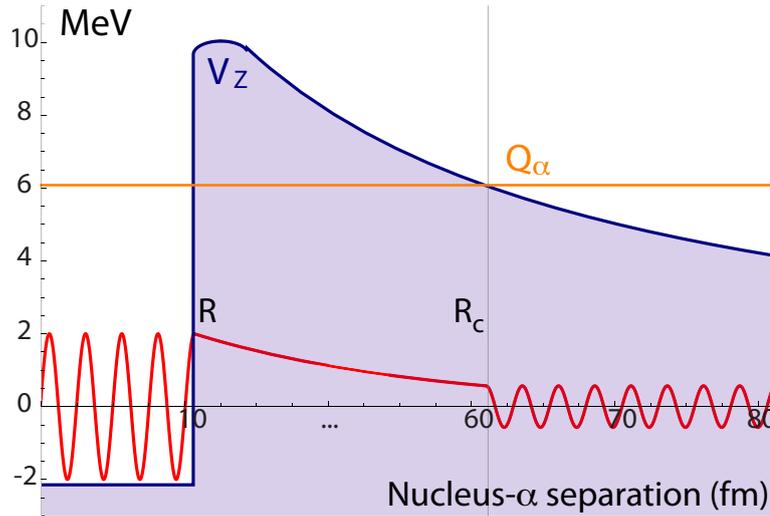
Fig. 17: Potential well for alpha decay tunneling. The inner radius is R while the intersection of Q_α with the potential is R_c (not to scale).
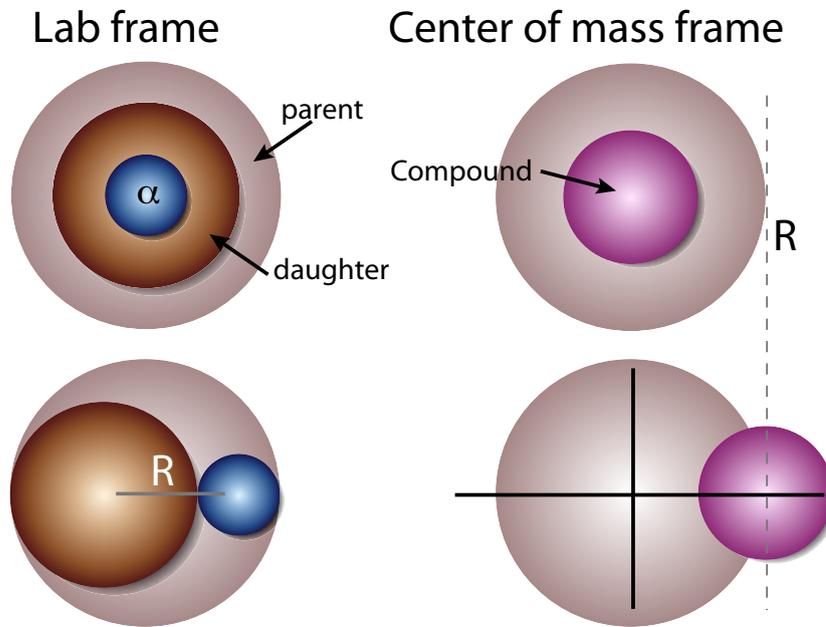

Fig. 18: Positions of daughter and alpha particles in the nucleus, as seen in (left) the laboratory frame and (right) in the center of mass frame. When the relative distance is zero, this correspond to a undivided (parent) nuclide. When the relative distance is R, it corresponds to a separate alpha and daughter nuclide inside the nucleus.

(see cartoon in figure 18). We have $\frac{1}{2}mv_{in}^2 = Q_\alpha + V_0 \approx 40\text{MeV}$, from which we have $v_{in} \approx 4 \times 10^{22}\text{fm/s}$. Then the frequency is $f \approx 4.3 \times 10^{21}$.

The probability of tunneling is given by the amplitude square of the wavefunction just outside the barrier, $P_T = |\psi(R_c)|^2$, where R_c is the coordinate at which $V_{Coul}(R_c) = Q_\alpha$, such that the particle has again a positive kinetic energy:

$$R_c = \frac{e^2 Z_\alpha Z'}{Q_\alpha} \approx 63\text{fm}$$

Recall that in the case of a square barrier, we expressed the wavefunction inside a barrier (in the classically forbidden region) as a plane wave with imaginary momentum, hence a decaying exponential $\psi_{in}(r) \sim e^{-\kappa r}$. What is the relevant momentum $\hbar\kappa$ here? Since the potential is no longer a square barrier, we expect the momentum (and kinetic energy) to be a function of position.

The total energy is given by $E = Q_\alpha$ and is the sum of the potential (Coulomb) and kinetic energy. As we've seen that the Coulomb energy is higher than Q, we know that the kinetic energy is negative:

$$Q_\alpha = T + V_{Coul} = \frac{\hbar^2 k^2}{2\mu} + \frac{Z_\alpha Z' e^2}{r}$$

with μ the reduced mass

$$\mu = \frac{m_\alpha m'}{m_\alpha + m'}$$

and $k^2 = -\kappa^2$ (with $\kappa \in R$). This equation is valid at any position inside the barrier:

$$\kappa(r) = \sqrt{\frac{2\mu}{\hbar^2} [V_{Coul}(r) - Q_\alpha]} = \sqrt{\frac{2\mu}{\hbar^2} \left(\frac{Z_\alpha Z' e^2}{r} - Q_\alpha \right)}$$

If we were to consider a small slice of the barrier, from r to $r + dr$, then the probability to pass through this barrier would be $dP_T(r) = e^{-2\kappa(r)dr}$. If we divide then the total barrier range into small slices, the final probability is the product of the probabilities dP_T^k of passing through all of the slices. Then $\log(P_T) = \sum_k \log(dP_T^k)$ and taking the continuous limit $\log(P_T) = \int_R^{R_c} \log[dP_T(r)] = -2\int_R^{R_c} \kappa(r)dr$.

Finally the probability of tunneling is given by $P_T = e^{-2G}$, where G is calculated from the integral

$$G = \int_R^{R_C} dr\kappa(r) = \int_R^{R_C} dr\sqrt{\frac{2\mu}{\hbar^2}\left(\frac{Z_\alpha Z' e^2}{r} - Q_\alpha\right)}$$

We can solve the integral analytically, by letting $r = R_c y = y\frac{Z_\alpha Z' e^2}{Q_\alpha}$, then

$$G = \frac{Z_\alpha Z_0 e^2}{\hbar c}\sqrt{\frac{2\mu c^2}{Q_\alpha}} \int_{R/R_C}^1 dy\sqrt{\frac{1}{y} - 1}$$

which yields

$$G = \frac{Z_\alpha Z' e^2}{\hbar c}\sqrt{\frac{2\mu c^2}{Q_\alpha}} \left[\arccos\left(\sqrt{\frac{R}{R_c}}\right) - \sqrt{\frac{R}{R_c}}\sqrt{1 - \frac{R}{R_c}} \right] = \frac{Z_\alpha Z' e^2}{\hbar c}\sqrt{\frac{2\mu c^2}{Q_\alpha}}\frac{\pi}{2}g\left(\sqrt{\frac{R}{R_c}}\right)$$

where to simplify the notation we used the function

$$g(x) = \frac{2}{\pi}\left(\arccos(x) - x\sqrt{1 - x^2}\right).$$

Finally the decay rate is given by

$$\boxed{\lambda_\alpha = \frac{v_{in}}{R}e^{-2G}}$$

where G is the so-called Gamow factor.
In order to get some insight on the behavior of G we consider the approximation $R \ll R_c$:

$$G = \frac{1}{2}\sqrt{\frac{E_G}{Q_\alpha}}g\left(\sqrt{\frac{R}{R_c}}\right) \approx \frac{1}{2}\sqrt{\frac{E_G}{Q_\alpha}}\left[1 - \frac{4}{\pi}\sqrt{\frac{R}{R_c}}\right]$$

where E_G is the Gamow energy:

$$\boxed{E_G = \left(\frac{2\pi Z_\alpha Z e^2}{\hbar c}\right)^2 \frac{\mu c^2}{2}}$$

For example for the ^{238}U decay studied $E_G = 122,000$MeV (huge!) so that $\sqrt{E_G/Q_\alpha} = 171$ while $g\left(\sqrt{\frac{R}{R_c}}\right) \approx 0.518$. The exponent is thus a large number, giving a very low tunneling probably: $e^{-2G} = e^{-89} = 4 \times 10^{-39}$. Then, $\lambda_\alpha = 1.6 \times 10^{-17}$s or $t_{1/2} = 4.5 \times 10^9$ years, close to what observed.

These results finally give an answer to the questions we had regarding alpha decay. The decay probability has a very strong dependence on not only Q_α but also on $Z_1 Z_2$ (where Z_i are the number of protons in the two daughters). This leads to the following observations:

- Other types of decay are less likely, because the Coulomb energy would increase considerably, thus the barrier becomes too high to be overcome.

- The same is true for spontaneous fission, despite the fact that Q is much higher ($\sim 200\text{MeV}$).

- We thus find that alpha decay is the optimal mechanism. Still, it can happen only for $A \geq 200$ exactly because otherwise the tunneling probability is very small.

- The Geiger-Nuttall law is a direct consequence of the quantum tunneling theory. Also, the large variations of the decay rates with Q are a consequence of the exponential dependence on Q.

A final word of caution about the model: the semi-classical model used to describe the alpha decay gives quite accurate predictions of the decay rates over many order of magnitudes. However it is not to be taken as an indication that the parent nucleus is really already containing an alpha particle and a daughter nucleus (only, it behaves as if it were, as long as we calculate the alpha decay rates).

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22.02 Introduction to Applied Nuclear Physics
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