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Lectures Presented at the Nuclear Engineering Department of the Massachusetts Institute of Technology (MIT)

Scattering: Coherent and Incoherent (Interference effects)

(A) Coherent Elastic Scattering

1. Occurs for waves describing incident neutrons having the same energies and the same spin state

2. For a incident neutron characterized by e^{ikz} each nucleus in the target emits a scattered wave (*s-wave*, low energy)

$$\psi_{si} = -a_i \frac{e^{ik_i r_i}}{r_i}$$

(Relative coordinate systems attached to the i^{th} nucleus)

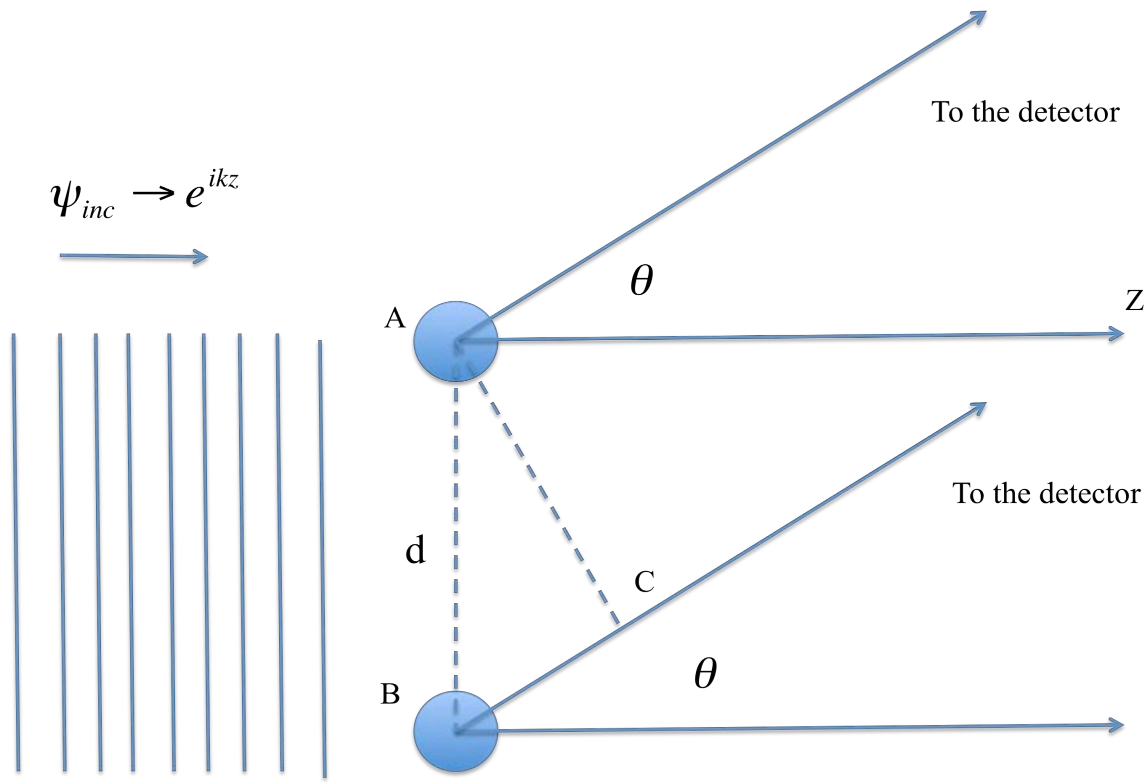
3. For N nuclei in the target, with same energy and spin in the laboratory system, the total wave (added coherently) is

$$\psi_s = \sum_{i=1}^{N} -a_i \frac{e^{ik_i \left|\vec{r}-\vec{R}_i\right|}}{\left|\vec{r}-\vec{R}_i\right|}$$

$\vec{R}_i$ Laboratory position of the i^{th} nucleus

4. Coherent scattering affects only the angular distribution of scattered neutrons. It does not affect the total scattering cross section (neutrons are conserved in the scattering)

Simple System



The scattered waves arriving at the detector are combinations of the waves scattered at A and B and are give by

$$\psi_{sc}(r,\theta,\varphi) \to -a\frac{e^{ikr}}{r} - a\frac{e^{ik(r+\overline{BC})}}{r+\overline{BC}}$$

The scattering lengths are the same for both nuclei.

Since $r >> \overline{BC}$, $r+\overline{BC}$ in the denominator can be replaced by r

The segment $\overline{BC}$ is give as

$$\overline{BC} = d\sin\theta\cos\varphi$$

Suggested Homework:
Show the above expression
Hint: Calculate the angle between two vectors in spherical coordinates.

The total scattered-wave function is

$$\psi_{sc}(r,\theta,\varphi) = -\frac{e^{ikr}}{r}\left[a + a\,e^{ikd\,\sin\theta\,\cos\varphi}\right]$$

Defining the scattering function as

$$f(\theta,\varphi) = \left[a + a\,e^{ikd\,\sin\theta\,\cos\varphi}\right]$$

The differential scattering cross section is

$$\sigma_s(\theta,\varphi) = \left| f(\theta,\varphi) \right|^2$$

The differential scattering cross section per nucleus for our simple system is obtained by using the expression for $f(\theta,\varphi)$ with a factor ½ (two nuclei)

$$\sigma_s(\theta,\varphi) = \frac{1}{2} a^2 \left[1 + \cos(kd \sin\theta \cos\varphi) \right]$$

Suggested homework:

Behavior of $\sigma_s(\theta,\varphi)$ **for** kd **<< 1 (low energy) and** kd **>> 1, (high energy)**

Consider the special case for $\varphi = 0$, i. e., $\cos\varphi = 1$

$$\sigma_s(\theta,0) = \frac{1}{2}a^2\left[1 + \cos(kd\sin\theta)\right]$$

Since $k = \frac{2\pi}{\lambda}$

$$\sigma_s(\theta,0) = \frac{1}{2}a^2\left[1 + \cos(2\pi\frac{d}{\lambda}\sin\theta)\right]$$

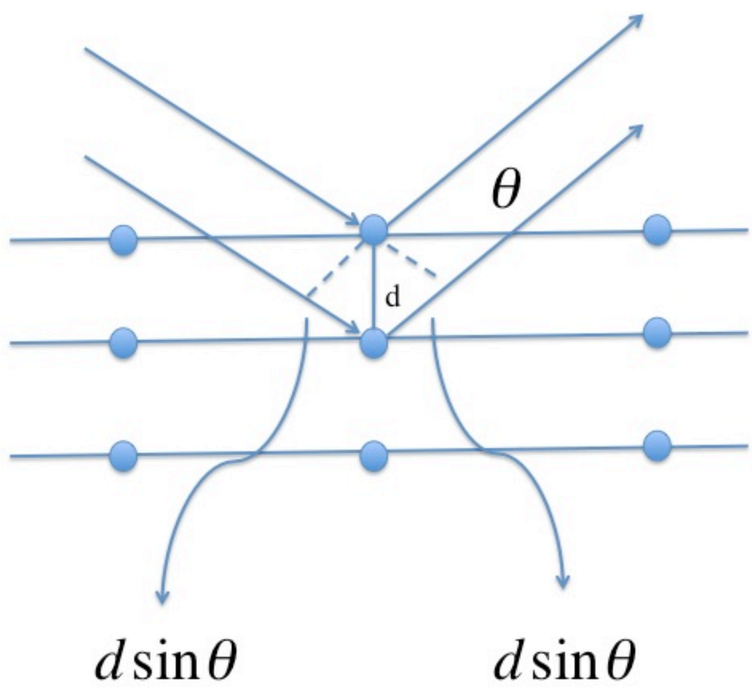
We see that the maximum of $\sigma_s(\theta,\varphi)$ occurs when

$$n\lambda = d\sin\theta$$

$$n = 1,2,3\ldots$$

This is the Bragg's law for our simple system.

In general the situation is:



Neutrons incident are scattered by an angle θ

Bragg's Law is

$$n\lambda = 2d\sin\theta$$

$$n = 1,2,3\ldots$$

Note that for:

(a) $\lambda > 2d$ no Bragg scattering is possible

(b) Maximum occur for $\lambda = 2d_{\max}$; this is the Bragg cutoff

We know that

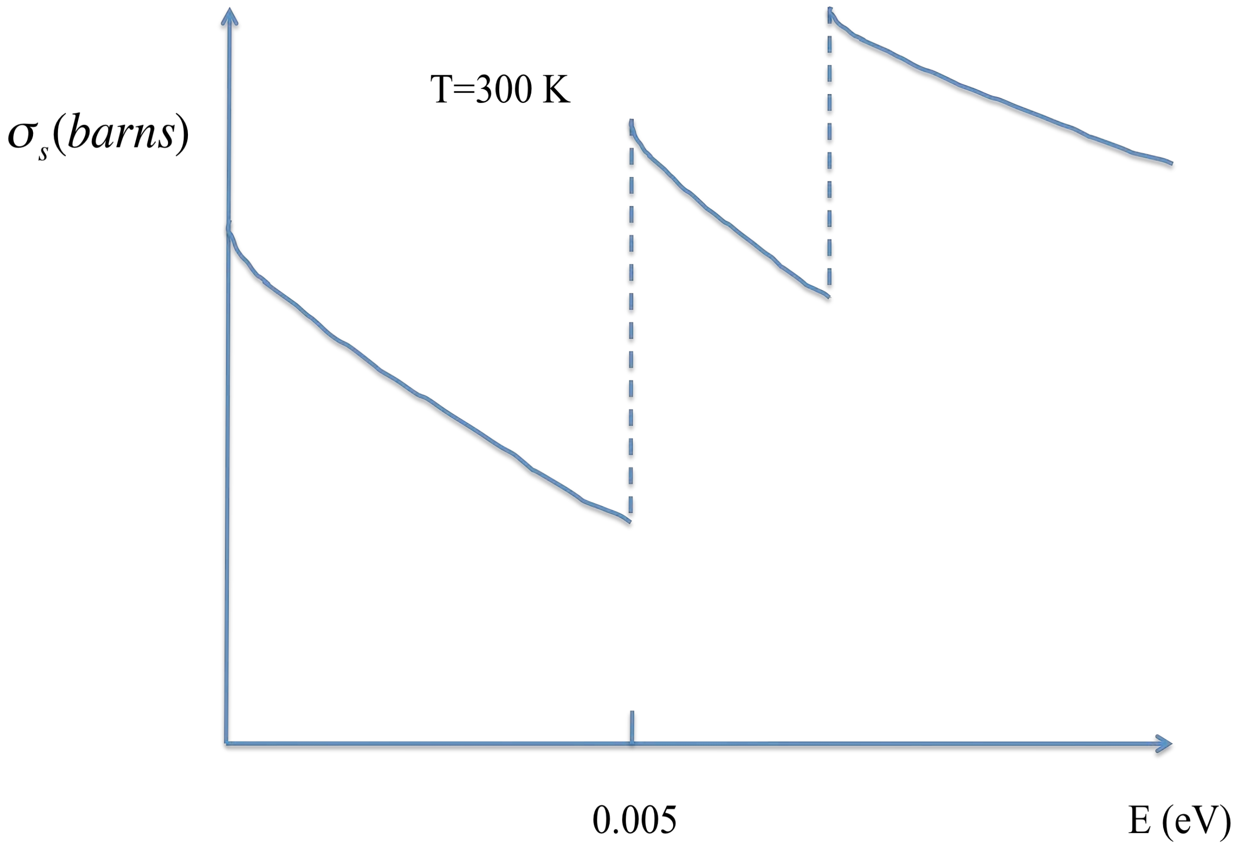
$$\lambda = \frac{2.86 \times 10^{-9}}{\sqrt{E(eV)}} \quad (cm)$$

Suggested homework:
Show that $\lambda = f(E)$ **as above**

The expression for the Bragg cutoff is

$$d_{\max} = \frac{0.143}{\sqrt{E(eV)}} \text{ (Å)}$$

1Å = $10^{-8}\,cm$



Calculate $d_{\max} = ?$

(B) Incoherent Elastic Scattering:

1. Scattering by non identical nuclei
2. Nuclear spin
3. Non-constant nuclear spacing (Crystal)

(a) Non-identical nuclei (continuing with the simple system)

In the simple system we have been studying for the scattering of two non-identical nuclei there will two distinct scattering lengths a_1 and a_2

The differential scattering cross section is

$$\sigma_s(\theta,\varphi) = \frac{1}{2}\left|a_1 + a_2 e^{ikd\,\sin\theta\,\cos\varphi}\right|^2$$

or

$$\sigma_s(\theta,\varphi) = \frac{1}{2}\left[a_1^2 + a_2^2 + 2a_1a_2\cos(kd\sin\theta\cos\varphi)\right]$$

Maximum:

$$\sigma_s^{\max}(\theta,\varphi) = \frac{1}{2}(a_1 + a_2)^2$$

Minimum:

$$\sigma_s^{\min}(\theta,\varphi) = \frac{1}{2}(a_1 - a_2)^2$$

If only coherent scattering is present $a_1 = a_2$ and $\sigma_s^{\min}(\theta,\varphi) = 0$.

The nonzero $\sigma_s^{\min}(\theta,\varphi) = 0$ indicates the presence of incoherent scattering.

In general for a number N of nuclei with scattering lengths $a_n \, (n = 1, 2, 3, \ldots, N)$ the differential scattering cross section per nuclei is:

$$\sigma_s(\theta,\varphi) = \frac{1}{N} \left| \sum_{n=1}^{N} a_n G_n(k, R_n) \right|^2$$

R_n Position of the nth nucleus

If G_n^* is the complex conjugate of G_n then $G_n G_n^* = G_n^* G_n = 1$

Some definitions:

$$< a > = \frac{1}{N} \sum_{n=1}^{N} a_n$$

The expression for $\sigma_s(\theta,\varphi)$ can be written as

$$\sigma_s(\theta,\varphi) = \frac{1}{N}\left|\sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m G_n G_m^*\right|$$

By using

$$a_n = (a_n - \langle a \rangle) + \langle a \rangle$$

we have

$$\sigma_s(\theta,\varphi) = \langle a^2 \rangle - \langle a \rangle^2 + \langle a \rangle^2 \frac{1}{N}\left|\sum_{n=1}^{N} a_n G_n\right|^2$$

Suggested homework:
Show the expression above

The scattering cross section is

$$\sigma_s = \int_{4\pi} \sigma_s(\theta,\varphi) d\Omega$$

Then

$$\sigma_s = 4\pi(<a^2> - <a>^2) + 4\pi <a>^2$$

The coherent and incoherent cross sections are:

i. Coherent

$$\sigma_s^{coh} = 4\pi <a>^2$$

ii. Incoherent

$$\sigma_s^{incoh} = 4\pi(<a^2> - <a>^2)$$

(b) Nuclear spin

Why is this a cause for interference??

Potential neutron-nucleus depends on the orientation of the spin vector.

$$V(r) \propto \vec{I}\,.\,\vec{i}$$

$$\vec{I} \rightarrow nuclear\ spin$$

$$\vec{i} \rightarrow neutron\, spin\ (1/2)$$

Channel spin: $\vec{J}$ is a sum of $\vec{I}$ and $\vec{i}$ that is $\vec{J} = \vec{I} + \vec{i}$

Therefore:

$$J^{+} = I + 1/2$$

and

$$J^{-} = I - 1/2$$

The statistical spin factor is given as

$$g(J) = \frac{2J+1}{(2i+1)(2I+1)}$$

For each J^+ and J^- we have

$$g^- = \frac{I}{2I+1}$$

and

$$g^+ = \frac{I+1}{2I+1}$$

The spin coherent and incoherent cross sections are

i. Spin coherent

$$\sigma_s^{coh} = 4\pi(g^+a_+^2 + g^-a_-^2)$$

ii. Spin incoherent

$$\sigma_s^{incoh} = 4\pi g^- g^+(a_+ - a_-)^2$$

Example for ^1H

$a_+ = 5.3\ fm$ $\qquad$ $a_+ = -24.0\ fm$

where $1\ fm = 10^{-13}\ cm$ and $I = 1/2$

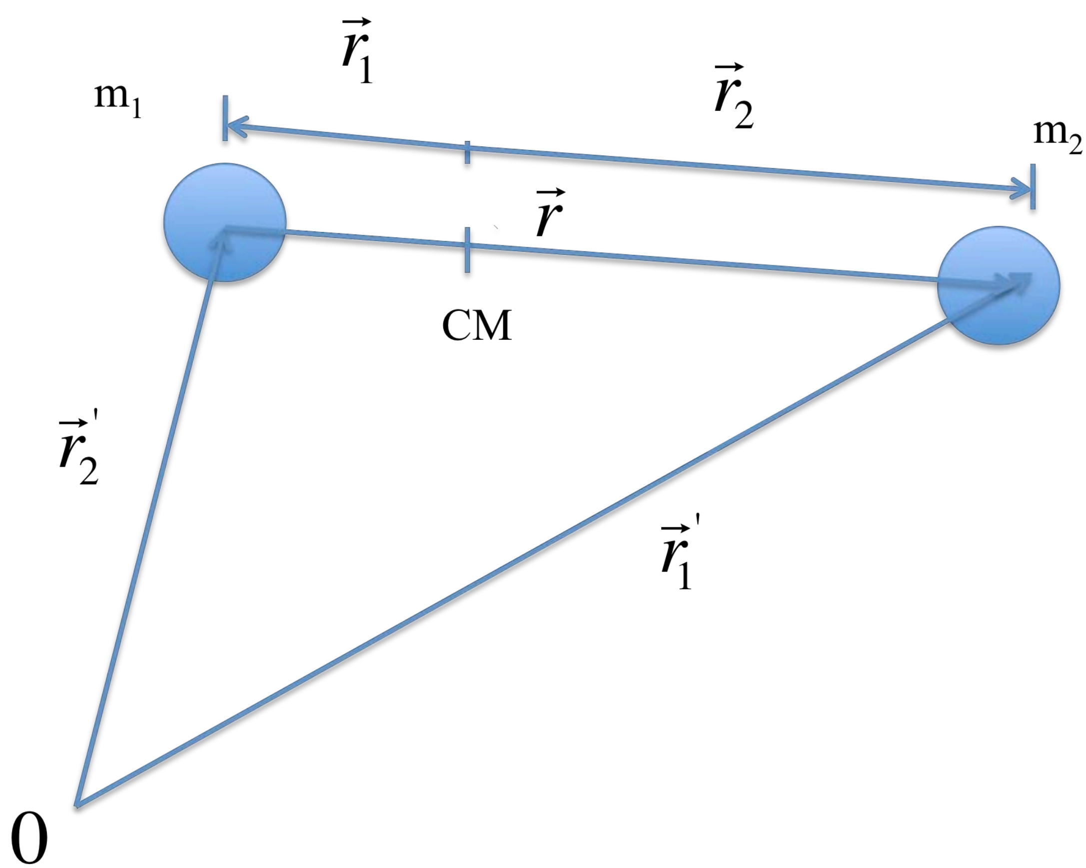
Calculate:

$\sigma_s^{coh} = ?$ $\qquad$ $\sigma_s^{incoh} = ?$ $\qquad$ $\sigma_s = ?$

Inelastic Effects

In the "Chemical energy regions" the notion of elastic and inelastic effects are related to the energy levels of the target. No change in the energy levels leads to an elastic scattering, otherwise, it is inelastic.

Let's examine the simple case of a diatomic molecular (Energy Levels). The possibilities are: **Translation, Vibration and Rotation**.



$$\vec{r} = \vec{r}_1 - \vec{r}_2$$

and

$$\vec{r} = \vec{r}_1^{'} - \vec{r}_2^{'}$$

The reduce mass is:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

(a) Vibrational mode:

Assume the mass m_1 and m_2 are bound together by an elastic spring such as

$$F = -k(x - x_0)$$

Classical | *Quantum – Mechanics*

$$\frac{d^2x}{dt^2} = -k(x - x_0)$$

$$x = x_0 \sin(wt + \varphi)$$

$$w = \sqrt{\frac{k}{m}}; \qquad w = 2\pi\nu$$

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E - \frac{1}{2}kx^2)\psi = 0$$

$$E_n = \hbar w(n + \frac{1}{2}); \quad n = 0,1,2,...$$

The energy levels will be

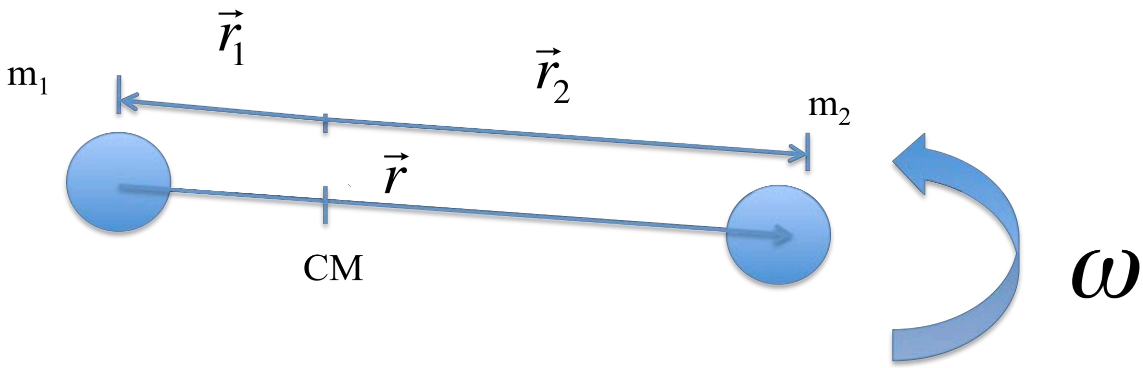
$$E_0 = \frac{1}{2}\hbar w$$

$$E_1 = \frac{3}{2}\hbar w$$

$$E_2 = \frac{5}{2}\hbar w$$

etc

(b) Rotational Mode:



(i) Classical energy of the system

$$E = \frac{1}{2}m_1\upsilon_1^2 + \frac{1}{2}m_2\upsilon_2^2$$

$$for$$

$$\upsilon_1 = wr_1 \quad and \quad \upsilon_2 = wr_2$$

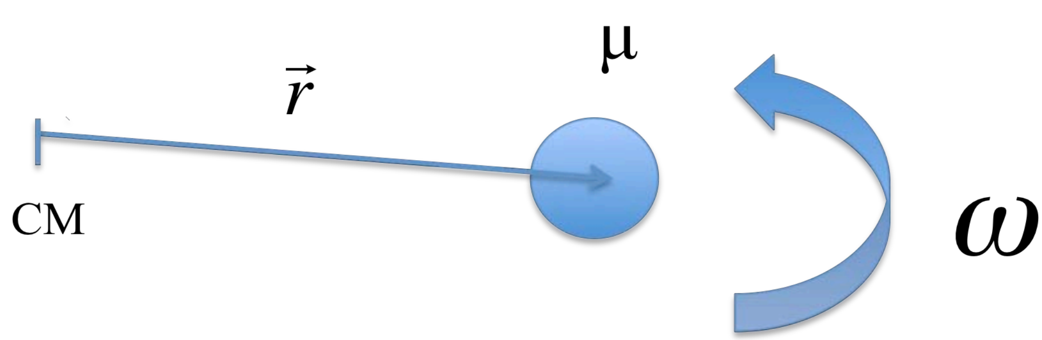
then

$$E = \frac{1}{2} I\, w^2 \quad where \quad I = m_1 r_1^2 + m_2 r_2^2$$

Using

$$\vec{r} = \vec{r}_1 + \vec{r}_2 \quad and \quad m_1 r_1 + m_2 r_2 = 0$$

The problem is reduced to a one-body problem



The angular momentum

$$L = mr\upsilon \quad or \quad L = mr^2 w \quad or \quad L = Iw$$

The rotation energy becomes

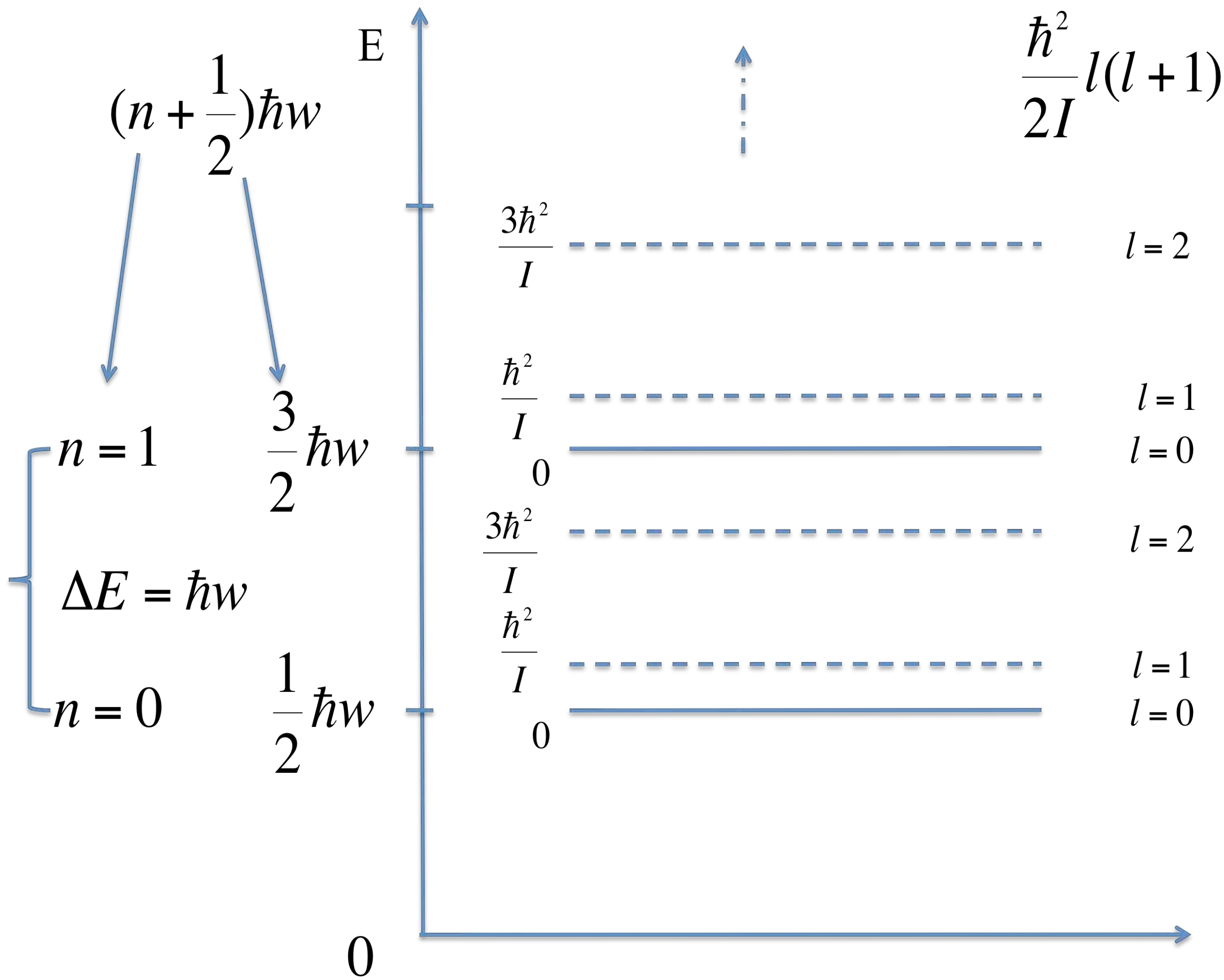
$$E = \frac{L^2}{2I}$$

(ii) Quantum-mechanics viewpoint

$$L^2 = \hbar^2\, l(l+1) \quad \textit{the energy will be:}$$

$$E = \frac{\hbar^2}{2I} l(l+1) \quad where \quad l = 0, 1, 2, \ldots$$

Vibrational-Rotational Energy Levels



In the inelastic scattering process the vibration mode of the system (a crystal for instance) may change due to a collision with neutrons. The vibrational energy quantized is called a **phonon**. In a more realistic system where several nuclei are present it is not possible to identify individual phonon. In that case one uses the concept of phonon spectrum (phonon distribution) usually a function of the frequency w $(2\pi v)$.

The energy of a phonon of frequency w is $\hbar w$ and the energy spectrum is $f(w)$

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