Exercise 2. Integrating Ordinary Differential Equations.

1. Reduce the following higher-order ordinary differential equations to first-order vector differential equations, which you should write out in vector format.
   (a) $\frac{d^2y}{dx^2} = -x$
   (b) $Ay + B\frac{dy}{dx} + C\frac{d^2y}{dx^2} + D\frac{d^3y}{dx^3} = E$
   (c) $(\frac{d^2y}{dx^2})^2 = 2(\frac{dy}{dx})^3 - y^2$

2. Accuracy order of ODE schemes. For notational convenience, we start at $x = y = 0$ and consider a small step in x and y of the ODE $dy/dx = f(y,x)$. The Taylor expansion of the derivative function along the orbit is

$$f(y(x), x) = f_0 + \frac{df_0}{dx}x + \frac{d^2 f_0}{dx^2}\frac{x^2}{2!} + \dots \tag{1}$$

   (a) Integrate $\frac{dy}{dx} = f(y, x)$ term by term to find the solution for $y(x)$ to third order in x.
   (b) Suppose $y_1 = f_0 h/2$,. Find $y_1 - y(h/2)$ to second order in h.
   (c) Now consider $y_2 = f(y_1, h/2)h$, show that it is equal to $y(h)$ plus a term that is third order in h; i.e. that it is accurate to second order.

[Comment. The third order term in part (c) involves the partial derivative $\partial f/\partial y$ rather than the derivative along the orbit. Proving rigorously that the fourth order Runge Kutta scheme really is fourth order, is rather difficult because it requires keeping track of such partial derivatives.]

3. A leap-frog scheme can be applied to an ordinary differential equation $dv/dt = F(x)$ and $dx/dt = v$. Suppose we have $x(t)$ and $v(t - \Delta t/2)$. We want to establish the order of accuracy of the leap-frog value of the next velocity

$$v(t + \Delta t/2) = v(t - \Delta t/2) + F(x(t))\Delta t.$$

The exact solution is $v(t + \Delta t/2) = v(t - \Delta t/2) + \int F(x)dt$. We want to evaluate it in the neighborhood of time t.
   Expand $x(t + \delta t)$ as $x = x(t) + v(t)\delta t + F(t)\delta t^2/2! + ...$
   Expand $F(x) = F(x(t)) + dF/dx \, \delta x + d^2 F/dx^2 \, \delta x^2/2! + ...$, where $\delta x = x - x(t)$.
(a) Substitute these into the integral and integrate term by term to evaluate

$$\Delta v = \int_{t-\Delta t/2}^{t+\Delta t/2} F(x)dt$$

to third order in Δt. [Remember that the integral introduces an additional factor of Δt so you only need to expand the integrand, $F(x)$, to second order.]
(b) Hence find the lowest order term of the error in Δv: namely $F(x(t))\Delta t - \int F dt$.
(c) If the time step range of the integral were instead asymmetric: $t - \Delta t_1 \to t + \Delta t_2$, where $\Delta t_1 \neq \Delta t_2$, what would be the accuracy then?

Programming Exercise (3) Write a program to integrate numerically from $t = 0$ to $t = t_{max}$ the ODE

$$\frac{dy}{dt} = \frac{-\omega y}{1 + y^2}$$

with ω a positive constant, starting from $y(0) = 1$, using the explicit Euler scheme

$$y_{n+1} = y_n - \Delta t \omega y_n / (1 + y_n^2).$$

(a) Find numerically the *fractional* error at $t = 4/\omega$ for the following choices of timestep (i) $\omega \Delta t = 0.01$ (ii) $\omega \Delta t = 0.1$ (iii) $\omega \Delta t = 1$.

(b) What do your results indicate is the order of accuracy of the scheme?

(c) Find experimentally the timestep value at which the scheme becomes unstable. (This might require you to use $t_{max} > 4/\omega$.) What is different about the long-time behavior of the unstable solution, from the case considered in the text?

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