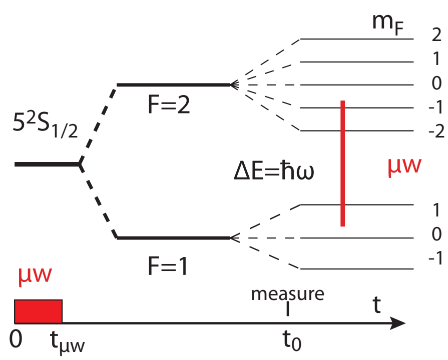
22.51 Quantum Theory of Radiation Interactions

Mid-Term Exam

October 31, 2012 Name: ....

In this mid-term we will study the dynamics of an atomic clock, which is one of the applications of David Wineland's research (Nobel prize in physics 2012). We consider a simplified (and sometime inaccurate) model of a vapor-cell atomic clock[1].

Light from a lamp passes through a vapor of rubidium atoms housed in a glass cell and is detected by a photodiode. The light intensity transmitted by the vapor is used to lock the frequency of an RF signal to an atomic transition. The atomic resonance that forms the basis of the Rubidium clocks operation is the transition between two hyperfine states $F = 1, 2$ of ^{87}Rb (see figure).



Problem 1: Rubidium Atomic Clock *30 points*

a) *Specifically, we monitor the frequency ω, corresponding to the energy difference between the Zeeman sub-levels $m_F = 0$ of the hyperfine angular momentum levels $F = 1, 2$.*

Formulate the simplest model that describes this atomic clock: What is the Hilbert space? What is a good basis? What is the Hamiltonian of interest?

Solution:

We can identify $|0\rangle = |F = 1, m_F = 0\rangle$ and $|1\rangle = |F=2, m_F=0\rangle$ as the basis for a two-level Hilbert space, with Hamiltonian:

$$\mathcal{H}_0 = \hbar\omega\sigma_z/2$$

I will in the following assume $\hbar = 1$.

b) *The atom is assumed to be at time $t = 0$ in the state $|F=2, m_F=0\rangle$. We turn on a transverse microwave field $B_y(t) = \Omega\cos(\omega t)$ (on resonance with the $|F=1, m_F=0\rangle \leftrightarrow |F=2, m_F=0\rangle$ transition).*

What is the total Hamiltonian?

The field is turned on only for a time for a time $t_{\mu w} = \pi/(2\Omega)$. Assuming $\Omega \ll \omega$, what is the state of the atom at $t = t_{\mu w}$?

Solution:

The microwave is represented by the Hamiltonian

$$\mathcal{H}_{\mu w} = \Omega\sigma_y\cos(\omega t)$$

thus the total Hamiltonian in the Schrödinger (or laboratory) picture is

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mu w} = \frac{\omega}{2}\sigma_z + \Omega\sigma_y\cos(\omega t)$$

[1] For a more accurate description, see e.g. J. Camparo, "The Rubidium Atomic Clock and Basic Research", Phys. Today 60, 33-39 (2007)

We can move into an interaction picture defined by $\mathcal{H}_0$. Then the Hamiltonian is given by

$$\mathcal{H}_I = \frac{\Omega}{2}\sigma_y + \frac{\Omega}{2}\sigma_y \cos(2\omega t) \approx \frac{\Omega}{2}\sigma_y$$

Thus, taking the rotating wave approximation, since $\Omega \ll \omega$, the total Hamiltonian in the interaction picture reduces to $\mathcal{H} = \frac{\Omega}{2}\sigma_y$.

Evolving the state $|1\rangle$ for a time $\pi/(2\Omega)$, we have $|\psi\rangle_I = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$. In the laboratory (or Schrödinger) picture this corresponds to

$$|\psi\rangle_S = e^{-i\omega t_{\mu w}\sigma_z/2}|\psi\rangle_I = \frac{1}{\sqrt{2}}(e^{i\omega t_{\mu w}/2}|1\rangle - e^{-i\omega t_{\mu w}/2}|0\rangle)$$

c) *At time $t = t_{\mu w}$ we turn off the microwave and let the atom evolve under its internal Hamiltonian for a time $t = t_0 - t_{\mu w}$. We then measure the operator M:*

$$M = |F\!=\!1, m_F\!=\!0\rangle\langle F\!=\!2, m_F\!=\!0| + |F\!=\!2, m_F\!=\!0\rangle\langle F\!=\!1, m_F\!=\!0|\,.$$

What is the signal we acquire at time t_0?

Solution:

The state at $t_0 - t_{\mu w}$ is

$$|\psi(t_0)\rangle = e^{-i\omega(t_0 - t_{\mu w}/2)\sigma_z}|\psi\rangle_S = \frac{1}{\sqrt{2}}(e^{i\omega t_0/2}|1\rangle - e^{-i\omega t_0/2}|0\rangle)$$

What we measure is $\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$, yielding $S(t_0) = -\cos(\omega t_0)$.

Note that if the measurement is also applied in the interaction frame (and this is indeed the usual case) the signal would have been $S(t_0) = -1$ (unless the pulse was applied off-resonance).

Problem 2: Optical pumping *40 points*

a) *We assumed above that the initial state of the atom was $|F = 2, m_F = 0\rangle$. In reality we have an ensemble of atoms at thermal equilibrium. Specifically, their kinetic energy corresponds to $k_BT \approx 40$meV, while the energy difference between the two levels of interest is $\Delta E \approx 20\mu$eV. What is the initial state of the atomic ensemble?*

[Write a formal expression for the state and then take the first order approximation in $\epsilon = \Delta E/(k_BT)$]

Solution:

Since we're at thermal equilibrium, it is represented by a mixed state in the macrocanonical ensemble:

$$\rho = \frac{1}{Z}e^{-\beta\mathcal{H}} = \frac{1}{Z}e^{-\frac{\Delta E}{k_BT}\frac{\sigma_z}{2}} \approx \frac{1}{2}(\mathbb{1} - \frac{\epsilon}{2}\sigma_z)$$

b) *What would be the signal in this case?*

Solution:

We can follow the same evolution as before. The signal was

$$S(t_0) = \langle\psi|\sigma_x|\psi\rangle = \text{Tr}\{\sigma_x|\psi\rangle\langle\psi|\}$$

Now we just have to replace $|\psi\rangle\langle\psi|$ with the new state. We calculated that the microwave irradiation corresponds to a $\pi/2$ rotation about the y axis. Thus we have at $t = t_{\mu w}$

$$\rho_I = \frac{1}{2}(\mathbb{1} - \frac{\epsilon}{2}\sigma_x) \qquad \rightarrow \qquad \rho_S = e^{-i\omega t_{\mu w}\sigma_z/2}\rho_I e^{i\omega t_{\mu w}\sigma_z/2} = \frac{1}{2}(\mathbb{1} - \frac{\epsilon}{2}[\sigma_x\cos(\omega t_{\mu w}) + \sigma_y\sin(\omega t_{\mu w})]),$$

where I used the well known rotation about the z-axis of σ_x.

Since the identity is not contributing to the signal, we have $S(t_0, \epsilon) = -\frac{\epsilon}{2}\cos(\omega t_0)$. The signal is thus very small, as ϵ is small.

c) *Because for a thermal state the signal is very small, we want to polarize the atomic ensemble. To do so, we send a continuous stream of light into the atomic ensemble.*
The light is thus the "environment" with ground state $|1\rangle$ (one photon). When the atom interacts with the light, it can absorb one photon with probability p, only if the atom is in the state $|F=1, m_F=0\rangle$ (and causing a transition to the state $|F=2, m_F=0\rangle$).
Write Kraus operators describing this process.

Solution:

The process described is the following:

$$U|11\rangle = |11\rangle$$
$$U|01\rangle = \sqrt{p}|10\rangle + \sqrt{1-p}|01\rangle$$

Thus the Kraus operators are

$$M_1 = |1\rangle\langle 1| + \sqrt{1-p}|0\rangle\langle 0|, \qquad M_2 = \sqrt{p}|1\rangle\langle 0|$$

d) *If the initial, thermal state is what you found in question (a), what is the state after applying once this process?*

Solution:

We can write the state as $\rho = \frac{1-\epsilon'}{2}|0\rangle\langle 0| + \frac{1+\epsilon'}{2}|1\rangle\langle 1|$. Applying the Kraus sum, we have:

$$\rho' = \frac{1-\epsilon'}{2}(1-p)|0\rangle\langle 0| + [\frac{1+\epsilon'}{2} + p\frac{1-\epsilon'}{2}]|1\rangle\langle 1| = \frac{1-\epsilon'}{2}(1-p)|0\rangle\langle 0| + [1 - \frac{1-\epsilon'}{2}(1-p)]|1\rangle\langle 1|$$

Thus, while the population in the ground state decreases, the population in the excited state increases: indeed, the absorption of photons create a population inversion, with a higher probability of the atom to be in the excited state.

e) *Assuming that the probability of photon absorption in a small time δt is $\delta p = \Gamma\delta t$, what is the state at a time $t_n = n\delta t$?*

Solution:

From the expression above we see that at each application of the Kraus map, the population of the zero state reduces by a factor $1-p$. Thus after n repetitions of the map, we expect to have the state

$$\rho_n = (1-\delta p)^n \frac{1-\epsilon'}{2}|0\rangle\langle 0| + [1 - (1-\delta p)^n \frac{1-\epsilon'}{2}]|1\rangle\langle 1|,$$

where I used the requirement of having trace 1 to simplify the calculations. Using the definitions of δp and n given above, we have

$$\rho_n = (1-\delta t\Gamma)^{t_n/\delta t} \frac{1-\epsilon'}{2}|0\rangle\langle 0| + [1 - (1-\delta t\Gamma)^{t_n/\delta t} \frac{1+\epsilon'}{2}]|1\rangle\langle 1|,$$

which for $\delta t \to 0$ gives an exponential decay, with a state

$$\rho(t_n) = \frac{1-\epsilon'}{2}e^{-\Gamma t_n}|0\rangle\langle 0| + [1 - \frac{1+\epsilon'}{2}e^{-\Gamma t_n}]|1\rangle\langle 1|$$

f) *What would be the signal if we take this as the initial state before performing the evolution described in the previous question?*

Solution:

We can rewrite the state found above as

$$\rho(t_n) = \frac{\mathbb{1}}{2} - \frac{1}{2}[1 - (1 - \frac{\epsilon}{2})e^{-\Gamma t_n}]\sigma_z$$

Thus we have a signal as calculated in Question b, but replacing $\epsilon/2$ with the value $1 - (1-\frac{\epsilon}{2})e^{-\Gamma t_n}$,

$$S(t_0, t_n) = -\left[1 - \left(1 - \frac{\epsilon}{2}\right)e^{-\Gamma t_n}\right]\cos(2\omega t_0).$$

For $t_n \to \infty$ we recover the signal $S(t_0, t_n) \approx -\cos(2\omega t_0)$ that we had for an initial pure state.

Problem 3: Clock Accuracy Limit *30 points*

We go back to the evolution described in Problem 1, but we now consider a more realistic system. Specifically, we take into account that the atoms collide with the wall of the glass cell. Each collision randomizes the phase of the atomic state.

a) *We can model the collisions as causing a "jump" of the phase described by the operator*

$$\sqrt{\Gamma_2}\,|F=2, m_F=0\rangle\langle F=2, m_F=0|\,.$$

Write a differential equation describing the system's evolution between time $t_{\mu w}$ and t_0.
Under which assumption(s) is this equation valid?

Solution:

The jump corresponds to a Lindblad operator $L = \sqrt{\Gamma}_2|1\rangle\langle 1|$. We have the Lindblad equation

$$\frac{d\,\rho}{d\,t} = i[\mathcal{H}_0, \rho] - L\rho L^\dagger + \frac{1}{2}(L^\dagger L\rho + \rho L^\dagger L)$$

$$\frac{d\,\rho}{d\,t} = i[\mathcal{H}_0, \rho] - \Gamma_2(\langle 1|\rho|1\rangle|1\rangle\langle 1| - \frac{1}{2}\rho|1\rangle\langle 1| - \frac{1}{2}|1\rangle\langle 1|\rho) = i\omega\begin{pmatrix} 0 & \langle 0|\rho|1\rangle \\ -\langle 1|\rho|0\rangle & 0 \end{pmatrix} - \Gamma_2\begin{pmatrix} 0 & \langle 0|\rho|1\rangle \\ \langle 1|\rho|0\rangle & 0 \end{pmatrix}$$

We can write a Lindblad equation only if the noise process is Markovian (memory-less). For this particular physical model this means that in each collision with the wall the phase is changed at random and there is no correlation between the phase change in different collisions.

b) *Assume that $t_{\mu w}\Gamma_2 \ll 1$, so that there are no collision during the rf pulse time (and we can consider the same evolution as in Problem 1 and the same state at $t = t_{\mu w}$). What is the state at time t_0? What is the signal?*

Solution:

The state we had found at time $t = t_{\mu w}$ was $|\psi\rangle_I = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$ or $|\psi\rangle_S = \frac{1}{\sqrt{2}}(e^{i\omega t_{\mu w}/2}|1\rangle - e^{-i\omega t_{\mu w}/2}|0\rangle)$. We study the evolution under $\mathcal{H}_0$ and the collisions of this state,

$$\rho(t_{\mu w}) = \frac{1}{2}\begin{pmatrix} 1 & -e^{-i\omega t_{\mu w}/2} \\ -e^{i\omega t_{\mu w}/2} & 1 \end{pmatrix}$$

From the Lindblad differential equation above, we have

$$\begin{cases} \dot\rho_{00} = \dot\rho_{11} = 0 \\ \dot\rho_{01} = (i\omega - \Gamma)\rho_{01} \\ \dot\rho_{10} = -(i\omega + \Gamma)\rho_{10}, \end{cases}$$

where I defined $\rho_{ij} = \langle i|\,\rho\,|j\rangle$. Thus at a time t_0 we have

$$\rho(t_{\mu w}) = \frac{1}{2}\begin{pmatrix} 1 & -e^{-i\omega t_0/2}e^{-\Gamma_2 t_0} \\ -e^{i\omega t_0/2}e^{-\Gamma_2 t_0} & 1 \end{pmatrix} = \frac{1}{2}\left(\mathbb{1} - [\sigma_x\cos(\omega t_0) + \sigma_y\sin(\omega t_0)]e^{-\Gamma_2 t_0}\right)$$

[Note that more precisely I should have written $e^{-\Gamma_2(t_0 - t_{\mu w})}$ but as assumed above I neglect the term $t_{\mu w}\Gamma_2 \ll 1$]

c) *A measure of the accuracy of a clock is the ratio of its signal mean frequency ν to its spread in frequency $\Delta\nu$, $Q = \frac{\nu}{\Delta\nu}$. Given the Fourier transform of the signal you found above, what is Q? What constraint does this impose on the rate of collision?*

Solution:

The signal we would measure in the presence of collisions is $S(t_0) = -\cos(\omega t_0)e^{-\Gamma_2 t_0}$, with Fourier Transform a Lorenztian centered around ω_0 and width Γ_2. We thus have $\nu = \omega$ and $\Delta\nu = \Gamma_2$. Thus we want the rate of collision to be much smaller than the clock frequency. This is obtained using a buffer gas.